**University of Illinois** Spring 2008

# ECE 313: Final Exam

Friday May 2, 2008

1. **[48 points, 4 per answer]**
In order to discourage guessing, 4 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score on this problem will reduce your total exam score.

   You do not need to show any work to justify your answers for *this* problem.

   (a) $A$ and $B$ are two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.

   Mark TRUE or FALSE for each question below.

| TRUE | FALSE | |
|---|---|---|
| □ | □ | $P(A \cup B) \geq \max\{P(A), P(B)\}$ |
| □ | □ | $P(A \cap B) \geq \min\{P(A), P(B)\}$ |
| □ | □ | $P(A\|B) + P(A\|B^c) = 1$. |
| □ | □ | $P(A\|B)P(B) + P(A^c\|B^c)P(B^c) = 1 - P(A \oplus B)$. |
| □ | □ | $P(A\|B)P(B) + P(A^c\|B)P(B) = P(A)$. |
| □ | □ | If $P(A) = P(B)$, then $P(A\|B) = P(B\|A)$. |
| □ | □ | If $P(A\|B) = P(B\|A)$, then $P(A) = P(B)$. |
| □ | □ | If $P(A\|B) = P(A)$, then $P(B^c\|A) = 1 - P(B)$. |

   (b) $\mathcal{X}$ and $\mathcal{Y}$ are random variables such that $\mathsf{var}(\mathcal{X}) = \mathsf{var}(\mathcal{Y}) = \sigma^2 < \infty$.
   Suppose that $\mathsf{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = \mathsf{var}(3\mathcal{X} - 2\mathcal{Y} + 1)$.
   Mark TRUE or FALSE for each of the following statements.

| TRUE | FALSE | |
|---|---|---|
| □ | □ | $\mathcal{X}$ and $\mathcal{Y}$ are *uncorrelated* random variables. |
| □ | □ | $\mathcal{X}$ and $\mathcal{Y}$ are *independent* random variables. |
| □ | □ | $\mathsf{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = \mathsf{var}(2\mathcal{X} - 3\mathcal{Y} + 1)$. |
| □ | □ | $\mathsf{cov}(\mathcal{X} + \mathcal{Y}, \mathcal{X} - \mathcal{Y}) = 0$. |

2. **[42 points]** At the Democratic National Convention (DNC), Hillary Clinton and Barack Obama have equal numbers of delegates committed to them, and neither candidate can win the nomination on a ballot. In desperation, the DNC decides to have a *series of debates* between the candidiates to decide the Democratic nominee. Hillary wins a debate (event $H$) with probability $p$, and Barack wins a debate (event $B$) with probability $q = 1 - p$. *There are no draws.* The debates continue until one of the candidates wins *two debates in a row* and is declared the Democratic nominee. Successive debates can be regarded as independent trials of an experiment, and $\mathcal{X}$ denotes the total number of debates.

   Express the answers to the following questions in terms of $p$ and $q$, that is, do not write (say) $pq$ as $p(1 - p)$ or multiply it out as $p - p^2$. On the other hand, feel free to simplify $p + q = 1$.

(a) **[6 points]** For $n > 0$, find the probability that *more than* $2n$ debates occur at the DNC.

(b) **[6 points]** For $n \geq 0$, find the probability that *more than* $2n+1$ debates occur at the DNC.

(c) **[12 points]** Find $\mathsf{E}[\mathcal{X}]$. Hint: use the results of parts (a) and (b).

(d) **[12 points]** Find $P\{\mathcal{X} = 2n+1\}$ and $P\{\mathcal{X} = 2n+1 \big| \mathcal{X} > 2n\}$.

Let $\bar{H}$ denote the event that Hillary wins the Democratic nomination. Note that this is not the same as the event $H$ that she wins a debate.

(e) **[6 points]** Find $P\{\bar{H} \big| \mathcal{X} = 2n+1\}$.

3. **[46 points]** The joint pdf of random variables $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} u+v, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) **[12 points]** Find the marginal pdf $f_{\mathcal{X}}(u)$ of the random variable $\mathcal{X}$. Be sure to specify the value of $f_{\mathcal{X}}(u)$ for all real numbers $u$.

(b) **[12 points]** Find the probability that the solutions of the quadratic equation $\alpha^2 + 2\mathcal{X}\alpha + \mathcal{Y} = 0$ are real numbers.

(c) **[12 points]** Find the conditional pdf $f_{\mathcal{Y}|\mathcal{X}}(v|\beta)$ of $\mathcal{Y}$ given that $\mathcal{X} = \beta$, where $0 < \beta < 1$. Be sure to specify the value of $f_{\mathcal{Y}|\mathcal{X}}(v|\beta)$ for all real numbers $v$.

(d) **[10 points]** Find the minimum-mean-square-error (MMSE) estimate of $\mathcal{Y}$ given that $\mathcal{X} = \beta$ where $0 < \beta < 1$.

4. **[34 points]** Consider the following binary hypothesis testing problem. If hypothesis $\mathsf{H}_0$ is true, the continuous random variable $\mathcal{X} \sim U(-2,2)$, while if hypothesis $\mathsf{H}_1$ is true, the pdf of $\mathcal{X}$ is $f_1(u) = \begin{cases} \frac{1}{4}(2 - |u|), & |u| < 2, \\ 0, & \text{otherwise.} \end{cases}$

(a) **[11 points]** The *maximum-likelihood* decision rule can be stated in the form $|\mathcal{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \eta$. Specify whether $x$ denotes 0 or 1, and find the values of $\eta$, the probability of false alarm $P_{\text{FA}}$, and the probability of missed detection $P_{\text{MD}}$.

(b) **[3 points]** Suppose that the hypotheses have *a priori* probabilities $\pi_0 = 1/3$ and $\pi_1 = 2/3$. What is the error probability $P(E)$ of the maximum-likelihood decision rule?

(c) **[14 points]** The MAP (also known as the minimum-error-probability or Bayesian) decision rule can be stated in the form $|\mathcal{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \xi$. Specify whether $x$ denotes 0 or 1, and find the values of $\xi$ and the error probability $P(E)$.

(d) [**3 points**] For what range (if any) of values of $\pi_0$, does the MAP decision rule always choose $\mathsf{H}_0$? If there are no such values of $\pi_0$, check this box $\square$.

(e) [**3 points**] For what range (if any) of values of $\pi_0$, does the MAP decision rule always choose $\mathsf{H}_1$? If there are no such values of $\pi_0$, check this box $\square$

5. [**30 points**] $\mathcal{X}$ and $\mathcal{Y}$ are independent random variables with pdfs as specified below:

$$f_{\mathcal{X}}(u) = \begin{cases} \exp(-u), & u > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{\mathcal{Y}}(v) = \begin{cases} \exp(v), & v < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{\mathcal{Z}}(\alpha)$, the pdf of the random variable $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$. Be sure to specify the value of $f_{\mathcal{Z}}(\alpha)$ for all real numbers $\alpha$.

6. [**25 points**] Suppose $\mathcal{X}$ and $\mathcal{Y}$ are jointly Gaussian random variables with means 0 and 2 respectively, variances 4 and 16 respectively, and correlation coefficient $\frac{1}{2}$. Let $\mathcal{W} = -\mathcal{X} + b\mathcal{Y}$ and $\mathcal{Z} = 7\mathcal{X} - \mathcal{Y}$ where $b$ is a number whose value you will determine below.

(a) [**5 points**] For what value(s) of $b$ does $\mathsf{E}[\mathcal{W}]$ equal 0?

(b) [**10 points**] For what value(s) of $b$ does $\mathsf{var}(\mathcal{W})$ equal 3?

(c) [**10 points**] For what value(s) of $b$ are $\mathcal{W}$ and $\mathcal{Z}$ independent random variables?