University of Illinois Spring 2008

ECE 313: Hour Exam II

Monday April 14, 2008
7:00 p.m. — 8:00 p.m.
Rooms 165 and 168 Everitt Laboratory

Use the supplied table of values of $\Phi(x)$, the unit Gaussian CDF, in solving this problem.

1. [33 points] The received signal $\mathcal{R}$ in a digital communication system corresponds to one of two hypotheses:

$$\begin{aligned} \mathsf{H}_0 &: \quad \mathcal{R} \sim \mathsf{Gaussian}(0, 2^2) \\ \mathsf{H}_1 &: \quad \mathcal{R} \sim \mathsf{Gaussian}(1, 2^2) \end{aligned}$$

(a) [10 points] The maximum-likelihood decision rule can be stated in the form

"Decide that H_1 is the true hypothesis if and only if $\mathcal{R} > \eta$."

What is the value of η, and what is the value of P_{FA}, the false-alarm probability (or probability of Type I error) of the maximum-likelihood decision rule?

(b) [18 points] Now suppose that the received signal $\mathcal{R}$ is passed through a quantizer whose output $\mathcal{S}$ is

$$\mathcal{S} = \begin{cases} -\alpha, & \text{if } \mathcal{R} < -1, \\ 0, & \text{if } -1 \leq \mathcal{R} < +1, \\ +\alpha, & \text{if } \mathcal{R} \geq +1. \end{cases}$$

and the decision is made based on the value of $\mathcal{S}$.
Complete the likelihood matrix shown below, and indicate on it the maximum-likelihood decision rule by shading entries.

	$\mathcal{S} = -\alpha$	$\mathcal{S} = 0$	$\mathcal{S} = +\alpha$
H_0			
H_1			

(c) [5 points] For the maximum-likelihood decision rule of part (b), what is P_{FA}?

2. [32 points]

   (a) [10 points] $\mathcal{W}$ denotes a *uniform* random variable with mean 1 and variance 3.

       Find $P\{\mathcal{W} < 0\}$.

   (b) [11 points] Suppose $\mathcal{X}$ is an exponential random variable with parameter λ.

       Calculate the pdf of the random variable $\mathcal{Y} = \sqrt{2\lambda\mathcal{X}}$.

   (c) [11 points] Suppose that $\mathcal{Z}$ is a standard Gaussian random variable. Find $\mathsf{E}[|\mathcal{Z}|]$.

3. [35 points] Consider a Poisson process with arrival rate λ. Let $N(a, b]$ denote the number of arrivals in the time interval $(a, b]$, and let $\mathcal{X}_1$ denote the time of the first arrival *after* $t = 0$.

   (a) [6 points] *State* the pmf of $N(0, \tau]$ and the pdf of $\mathcal{X}_1$. No derivation need be provided.

   (b) [12 points] The event $A = \{N(0, \tau] = 0\}$ is the same as the event $B = \{\mathcal{X}_1 > \tau\}$. *According to your answers of part (a),* what is $P\{N(0, \tau] = 0\}$? and what is $P\{\mathcal{X}_1 > \tau\}$? Does $P(A)$ equal $P(B)$?

   (c) [10 points] Now let T denote a fixed positive number and n a positive integer. Let τ be such that $0 < \tau < T$. If C denotes the event $\{N(0, T] = n\}$, what is the value of $P(B \middle| C)$?
       Hint: Express $P(B \cap C) = P(A \cap C)$ in terms of $N(0, \tau]$ and $N(\tau, T]$.

   (d) [7 points] Determine the *conditional* pdf of $\mathcal{X}_1$ given that the event C occurred, that is, given that n arrivals occurred in the time interval $(0, T]$.