

1. A linear minimum mean square error estimator

(a) Note that $\text{Cov}(Y, X) = \text{Var}(Y)$ and $\text{Var}(X) = \text{Var}(Y) + \text{Var}(N)$. By the formulas for linear MMSE estimation,

$$L = E[Y] + \frac{\text{Cov}(Y, X)}{\text{Var}(X)}(X - E[X]) = \frac{1}{\lambda} + \frac{1/\lambda^2}{1/\lambda^2 + \sigma^2}(X - \frac{1}{\lambda}) = \frac{1}{\lambda} + \frac{1}{1 + \lambda^2 \sigma^2}(X - \frac{1}{\lambda}) = \frac{X + \lambda \sigma^2}{1 + \lambda^2 \sigma^2}$$

$$E[e^2] = \text{Var}(Y) - \text{Cov}(Y, X)^2 / \text{Var}(X) = \frac{1}{\lambda^2} - (1/\lambda^4) / (\frac{1}{\lambda^2} + \sigma^2) = \frac{\sigma^2}{1 + \lambda^2 \sigma^2}$$

(b) Observe that Y is nonnegative, but the estimator L can be negative. An estimator with smaller MSE is $\hat{Y} = \max\{0, L\}$, because $(Y - \hat{Y})^2 \leq (Y - L)^2$ with probability one, and the inequality is strict whenever $L < 0$.

2. Working with joint Gaussians

Let $Z = X + 2Y$. Then Z is a linear combination of jointly Gaussian random variables, and so Z itself is a Gaussian random variable. Also, $E[Z] = E[X] + 2E[Y] = 0$ and

$$\begin{aligned} \sigma_Z^2 &= \text{Cov}(X + 2Y, X + 2Y) = \text{Cov}(X, X) + \text{Cov}(X, 2Y) + \text{Cov}(2Y, X) + \text{Cov}(2Y, 2Y) \\ &= \sigma_X^2 + 4\text{Cov}(X, Y) + 4\sigma_Y^2 = 5 - 4 + 8 = 9. \end{aligned}$$

Then, $P\{X + 2Y \geq 1\} = P\{Z \geq 1\} = P\{\frac{Z}{3} \geq \frac{1}{3}\} = Q(\frac{1}{3}) = 1 - \Phi(\frac{1}{3}) \approx 0.3693$

3. Simulating joint Gaussians

(a) Set $A = \sqrt{-\ln(U)} \cos(2\pi V)$ and $B = \sqrt{-\ln(U)} \sin(2\pi V)$. This is basically the representation of two independent unit normals in polar coordinates. The random variable $-\ln(U)$ is exponentially distributed with parameter one, and $\sqrt{-\ln(U)}$ has the Rayleigh density with parameter one. Also, $2\pi V$ is uniformly distributed over the interval $[0, 2\pi]$. For more information, see your notes from class, or lecture 37 on the course powerpoint slides

<http://courses.ece.uiuc.edu/ece413/spring07/Slides.html>

(b) The random variables X and Y defined in part (b) have mean zero, as required. So it is only necessary to match the variances and covariance given in the previous problem, yielding the conditions:

$$a^2 + b^2 = 5 \quad c^2 + d^2 = 2 \quad ac + bd = -1$$

There is a one-dimensional family of solutions, basically because there are four unknowns but only three equations to be solved. One solution is $(a, b, c, d) = (1, 2, 1, -1)$.

4. Estimating the square of a joint Gaussian

Recall the fact that $E[Z^2] = E[Z]^2 + \text{Var}(Z)$ for any second order random variable Z . The idea is to apply the fact to the conditional distribution of Y given X . Given $X = x$, the conditional distribution of X is Gaussian with mean ρx and variance $1 - \rho^2$. Thus, $E[Y^2|X = x] = (\rho x)^2 + 1 - \rho^2$. Therefore, $E[Y^2|X] = (\rho X)^2 + 1 - \rho^2$.

5. Weak law of large numbers for slightly correlated variables

(a) For any n ,

$$\begin{aligned}
 \text{Var}(S_n) &= \text{Cov}(X_1 + \cdots + X_n, X_1 + \cdots + X_n) \\
 &= \sum_{i=1}^n \sum_{j=1}^n \text{Cov}(X_i, X_j) \\
 &= \sum_{i=1}^{n-1} \text{Cov}(X_i, X_{i+1}) + \sum_{i=1}^n \text{Cov}(X_i, X_i) + \sum_{i=2}^n \text{Cov}(X_i, X_{i-1}) \\
 &\leq (n-1)(0.1) + n + (n-1)(0.1) < (1.2)n
 \end{aligned}$$

(b) Therefore, $\text{Var}(\frac{S_{100}}{100}) = \frac{1}{(100)^2} \text{Var}(S_{100}) \leq 0.012$. Also, $E[\frac{S_{100}}{100}] = \mu = 5$. Thus, by Chebychev's inequality, $P(|\frac{S_{100}}{100} - 5| \geq 0.5) \leq \frac{0.012}{(0.5)^2} = 0.048$.

6. Normal approximation for quantization error

The mean of each roundoff error is zero and the variance is $\int_{-0.5}^{0.5} u^2 du = \frac{1}{12}$. Thus, $E[S] = 0$ and $\text{Var}(S) = \frac{100}{12} = 8.333$. Thus, $P[|S| \geq 5] = P[|\frac{S}{\sqrt{8.333}}| \geq \frac{5}{\sqrt{8.333}}] \approx 2Q(\frac{5}{\sqrt{8.333}}) = 2Q(1.73) = 2(1 - \Phi(1.732)) = 0.083$.

7. A comparison of bounds and the normal approximation

(a) $E[X_1] = \frac{1+2+3+4+5+6}{6} = 3.5$ so by Markov's inequality, $P[S_{100} \geq 400] \leq \frac{100(3.5)}{400} = \frac{7}{8}$. (Not at all a strong bound.)

(b) $E[X_1^2] = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = 15$ so that $\text{Var}(X_i) = 15 - 3.5^2 = 2.75$ Therefore by Chebyshev's inequality $P[S_{100} \geq 400] = P[S_{100} - 350 \geq 50] \leq P[|S_{100} - 350| \geq 50] \leq \frac{100(2.75)}{50^2} = 0.11$

Or, we could use the one-side Chebyshev inequality found in Section 8.5 of Ross. That yields the somewhat tighter bound: $P[S_{100} \geq 400] = P[S_{100} - 350 \geq 50] \leq \frac{275}{275+2500} = 0.0991$. (Either answer is acceptable.) (c) The normal approximation based on the central limit theorem yields

$$P[S_{100} \geq 400] = P[\frac{S_{100} - 350}{\sqrt{(2.75)100}} \geq \frac{400 - 350}{\sqrt{(2.75)100}}] \approx Q(\frac{400 - 350}{\sqrt{(2.75)100}}) = Q(3.015) = 1 - \Phi(3.015) \approx 0.0009$$

8. A three state Markov chain

(a) $P = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.9 \end{pmatrix}.$

(b) The answer sought is $(P_{10}^{(4)}, P_{11}^{(4)}, P_{12}^{(4)}) = (0.2145, 0.5290, 0.2565)$. Though not required in the so-

lution, here are some powers of P : $P^2 = \begin{pmatrix} 0.6600 & 0.3200 & 0.0200 \\ 0.1600 & 0.6700 & 0.1700 \\ 0.0100 & 0.1700 & 0.8200 \end{pmatrix}$ $P^4 = \begin{pmatrix} 0.4870 & 0.4290 & 0.0840 \\ 0.2145 & 0.5290 & 0.2565 \\ 0.0420 & 0.2565 & 0.7015 \end{pmatrix}$
 $P^8 = \begin{pmatrix} 0.3327 & 0.4574 & 0.2099 \\ 0.2287 & 0.4377 & 0.3336 \\ 0.1049 & 0.3336 & 0.5614 \end{pmatrix}$ $P^{16} = \begin{pmatrix} 0.2373 & 0.4224 & 0.3403 \\ 0.2112 & 0.4075 & 0.3813 \\ 0.1701 & 0.3813 & 0.4485 \end{pmatrix}$ $P^{32} = \begin{pmatrix} 0.2034 & 0.4021 & 0.3945 \\ 0.2011 & 0.4007 & 0.3983 \\ 0.1972 & 0.3983 & 0.4045 \end{pmatrix}$

(c) The equilibrium conditions in matrix form are $\Pi = \Pi P$, or as separate equations are:

$$\begin{aligned} \Pi_0 &= \Pi_0 0.8 + \Pi_1 0.1 \\ \Pi_1 &= \Pi_0 0.2 + \Pi_1 0.8 + \Pi_2 0.1 \\ \Pi_2 &= \Pi_1 0.1 + \Pi_2 0.9. \end{aligned}$$

(Note: While there are three equations in three unknowns, the equations are dependent, because the sum of the equations is always true. However, another equation comes from the fact that the equilibrium probabilities sum to one.) The first and third equations yield $\Pi_0 = 0.5\Pi_1$ and $\Pi_2 = \Pi_1$ so that $\Pi = (0.5\Pi_1, \Pi_1, \Pi_1)$. Since the probabilities must sum to one, we can solve for Π_1 to get $\Pi = (0.2, 0.4, 0.4)$. All rows of P^n converge to Π as $n \rightarrow \infty$.