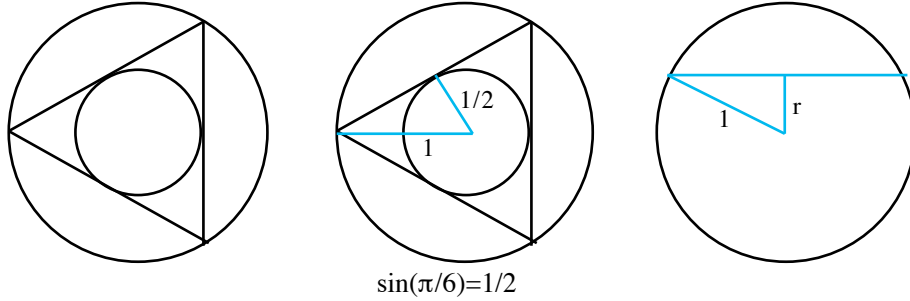


1. The pdf of $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the unit disc, and thus has value π^{-1} .



- (a) If the midpoint of the chord is at distance less than $1/2$ from the center of the circle, the chord is longer than the side of the equilateral triangle inscribed in the circle (cf. Ross, p. 217). Thus, the length $\mathcal{L}$ of the random chord is greater than the side of the equilateral triangle if $(\mathcal{X}, \mathcal{Y})$ lies inside a circle of radius $1/2$. This has probability $\pi^{-1} \left[\pi \left(\frac{1}{2} \right)^2 \right] = \frac{1}{4}$.
- (b) The length of a chord at distance r from the center is $2\sqrt{1-r^2}$.
Hence, $\mathcal{L} = 2\sqrt{1-\mathcal{R}^2} = 2\sqrt{1-\mathcal{X}^2-\mathcal{Y}^2}$
and $F_{\mathcal{L}}(\alpha) = P\{\mathcal{L} \leq \alpha\} = P\{\mathcal{X}^2 + \mathcal{Y}^2 \geq 1 - \alpha^2/4\} = 1 - \pi(1 - \alpha^2/4)\pi^{-1} = \alpha^2/4$. From this we get that $f_{\mathcal{L}}(\alpha) = \alpha/2$ for $0 \leq \alpha \leq 2$, and $f_{\mathcal{L}}(\alpha) = 0$ otherwise.
- (c) $E[\mathcal{L}] = \int_0^2 \alpha \cdot \frac{\alpha}{2} d\alpha = \frac{4}{3}$. Alternatively, LOTUS gives $E[\mathcal{L}] = E[2\sqrt{1-\mathcal{X}^2-\mathcal{Y}^2}]$

$$= \pi^{-1} \int \int 2\sqrt{1-u^2-v^2} du dv = \int_{r=0}^1 \int_{\theta=0}^{2\pi} \pi^{-1} 2\sqrt{1-r^2} r d\theta dr = \int_{r=0}^1 4r\sqrt{1-r^2} dr$$

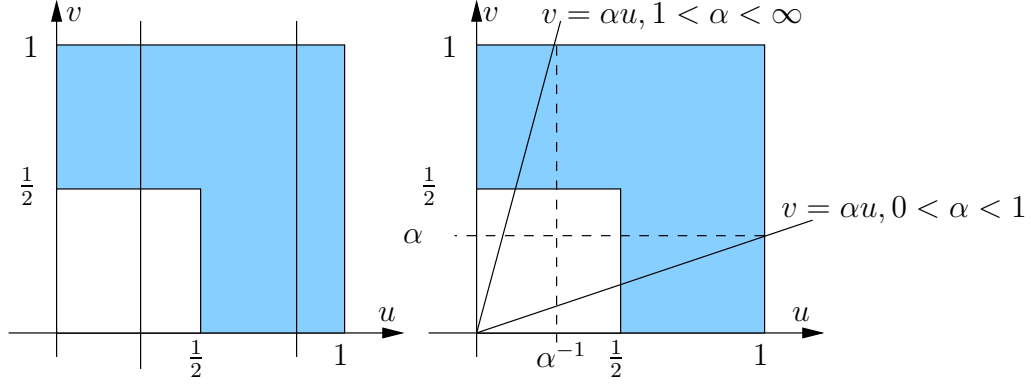
$$= -\frac{4}{3}(1-r^2)^{3/2} \Big|_0^1 = \frac{4}{3} \text{ as before.}$$
2. (a) The pdf surface is a right circular cone with base diameter 4 and altitude $2c$. The volume under the surface is $\frac{1}{3}\pi 2^2 2c = \frac{8\pi c}{3}$ giving $c = \frac{3}{8\pi}$. The mensurationally-challenged can use $1 = \int_{r=0}^2 \int_{\theta=0}^{2\pi} c(2-r) r d\theta dr = 2\pi c \left(r^2 - \frac{r^3}{3} \right) \Big|_0^2 = 8\pi c - \frac{16\pi c}{3} = \frac{8\pi c}{3}$ to arrive at the same result.
- (b) For $0 \leq \alpha \leq 2$, $F_{\mathcal{R}}(\alpha) = P\{\mathcal{R} \leq \alpha\} = \int_{r=0}^{\alpha} \int_{\theta=0}^{2\pi} \frac{3}{8\pi} (2-r) r d\theta dr = \frac{3}{4} \left(r^2 - \frac{r^3}{3} \right) \Big|_0^{\alpha}$

$$= \frac{3}{4} \left(\alpha^2 - \frac{\alpha^3}{3} \right)$$
 giving $f_{\mathcal{R}}(\alpha) = \frac{3}{2}\alpha - \frac{3}{4}\alpha^2$ for $0 \leq \alpha \leq 2$ and $f_{\mathcal{R}}(\alpha) = 0$ otherwise. It is also possible to find $F_{\mathcal{R}}(\alpha)$ directly by noting that the volume inside the circle of radius α is that of a cylinder of height $(3/8\pi)(2-\alpha)$ plus that of a cone of height $(3/8\pi)\alpha$ which volume is $\pi\alpha^2(3/8\pi)(2-\alpha) + (1/3)\pi\alpha^2(3/8\pi)\alpha = \frac{3}{4} \left(\alpha^2 - \frac{\alpha^3}{3} \right)$ as before.
3. (a) For $\alpha < \beta$, $F_{\mathcal{W}, \mathcal{Z}}(\alpha, \beta) = P\{\mathcal{W} \leq \alpha, \mathcal{Z} \leq \beta\} = P\{\max\{\mathcal{X}, \mathcal{Y}\} \leq \alpha, \min\{\mathcal{X}, \mathcal{Y}\} \leq \beta\}$

$$= P\{\max\{\mathcal{X}, \mathcal{Y}\} \leq \alpha, \min\{\mathcal{X}, \mathcal{Y}\} \leq \alpha\} = P\{\mathcal{X} \leq \alpha, \mathcal{Y} \leq \alpha\} = F_{\mathcal{X}, \mathcal{Y}}(\alpha, \alpha)$$
 since if the maximum is smaller than α which is itself smaller than β , then the minimum is also smaller than α .

- For $\alpha > \beta$, $F_{\mathcal{W}, \mathcal{Z}}(\alpha, \beta) = P\{\mathcal{W} \leq \alpha, \mathcal{Z} \leq \beta\} = P\{\max\{\mathcal{X}, \mathcal{Y}\} \leq \alpha, \min\{\mathcal{X}, \mathcal{Y}\} \leq \beta\}$
 $= P\{\mathcal{X} \leq \alpha, \mathcal{Y} \leq \beta\} + P\{\mathcal{X} \leq \beta, \mathcal{Y} \leq \alpha\} - P\{\mathcal{X} \leq \beta, \mathcal{Y} \leq \beta\}$
 $= F_{\mathcal{X}, \mathcal{Y}}(\alpha, \beta) + F_{\mathcal{X}, \mathcal{Y}}(\beta, \alpha) - F_{\mathcal{X}, \mathcal{Y}}(\beta, \beta).$
- (b) For $\alpha < \beta$, $F_{\mathcal{W}, \mathcal{Z}}(\alpha, \beta) = F_{\mathcal{X}, \mathcal{Y}}(\alpha, \alpha) = F_{\mathcal{X}}(\alpha)F_{\mathcal{Y}}(\alpha).$
For $\alpha > \beta$, $F_{\mathcal{W}, \mathcal{Z}}(\alpha, \beta) = F_{\mathcal{X}, \mathcal{Y}}(\alpha, \beta) + F_{\mathcal{X}, \mathcal{Y}}(\beta, \alpha) - F_{\mathcal{X}, \mathcal{Y}}(\beta, \beta)$
 $= F_{\mathcal{X}}(\alpha)F_{\mathcal{Y}}(\beta) + F_{\mathcal{X}}(\beta)F_{\mathcal{Y}}(\alpha) - F_{\mathcal{X}}(\beta)F_{\mathcal{Y}}(\beta).$
- (c) $f_{\mathcal{W}, \mathcal{Z}}(\alpha, \beta) = \frac{\partial^2}{\partial \alpha \partial \beta} F_{\mathcal{W}, \mathcal{Z}}(\alpha, \beta) = \begin{cases} f_{\mathcal{X}, \mathcal{Y}}(\alpha, \beta) + f_{\mathcal{X}, \mathcal{Y}}(\beta, \alpha), & \text{if } \alpha > \beta, \\ 0, & \text{if } \alpha < \beta. \end{cases}$
4. (a) For $v > 0$, $f_{\mathcal{X}^2}(v) = \frac{1}{2\sqrt{v}} \left[f_{\mathcal{X}}(\sqrt{v}) + f_{\mathcal{X}}(-\sqrt{v}) \right] = \frac{\exp(-u/2\sigma^2)}{\sigma\sqrt{2\pi v}} = \sqrt{\frac{\lambda}{\pi v}} \exp(-\lambda v)$
where $\lambda = \frac{1}{2\sigma^2}$ which is a gamma pdf with parameters $\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right).$
- (b) The pdfs of $\mathcal{X}^2$, $\mathcal{Y}^2$, and $\mathcal{Z}^2$ are all gamma pdfs with parameters $(\frac{1}{2}, \frac{1}{2\sigma^2}).$ The random variables $\mathcal{X}^2$, $\mathcal{Y}^2$, and $\mathcal{Z}^2$ are independent because $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are independent random variables.
- (c) The sum of independent gamma random variables with the same scale parameter is a gamma random variable with the same scale parameter, and with order parameter equal to the sum of the order parameters. Hence, the pdf of $\mathcal{W} = \mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2$ is a gamma pdf with parameters $(\frac{3}{2}, \frac{1}{2\sigma^2}),$ and is given by $f_{\mathcal{W}}(\alpha) = \frac{1}{\sigma^3} \sqrt{\frac{\alpha}{2\pi}} \exp(-\alpha/2\sigma^2)$ for $\alpha > 0.$
- (d) The expected value of a gamma random variable with parameters (t, λ) is $t/\lambda.$ Hence, $E[\mathcal{W}] = 3\sigma^2.$ Alternatively, $E[\mathcal{W}] = E[\mathcal{X}^2] + E[\mathcal{Y}^2] + E[\mathcal{Z}^2] = 3(\sigma^2 + (E[\mathcal{X}])^2) = 3\sigma^2.$
- (e) $E[\mathcal{H}] = \frac{1}{2}mE[\mathcal{W}] = \frac{3}{2}m\sigma^2 = \frac{3}{2}kT$ giving $\sigma^2 = kT/m.$ Hence, upon substitution, we get
 $f_{\mathcal{W}}(\alpha) = \left(\frac{kT}{m}\right)^{-3/2} \sqrt{\frac{\alpha}{2\pi}} \exp(-m\alpha/2kT)$ for $\alpha > 0$, while $\mathcal{H} = \frac{1}{2}m\mathcal{W}$ has pdf given by
 $f_{\mathcal{H}}(\beta) = \frac{2}{m} f_{\mathcal{W}}\left(\frac{2\beta}{m}\right) = \frac{2}{\sqrt{\pi}} (kT)^{-\frac{3}{2}} \sqrt{\beta} \exp(-\beta/kT)$ for $\beta > 0.$
- (f) $F_{\mathcal{V}}(\gamma) = P\{\mathcal{V} \leq \gamma\} = P\{\sqrt{\mathcal{W}} \leq \gamma\} = P\{\mathcal{W} \leq \gamma^2\} = F_{\mathcal{W}}(\gamma^2).$ Hence,
 $f_{\mathcal{V}}(\gamma) = \frac{d}{d\gamma} F_{\mathcal{V}}(\gamma) = 2\gamma f_{\mathcal{W}}(\gamma^2) = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT}\right)^{\frac{3}{2}} \gamma^2 \exp\left(-\frac{m\gamma^2}{2kT}\right), \gamma > 0.$ (cf. Theoretical Exercise 1 of Chapter 5).

5. The random point $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the shaded region shown in the figures below. Clearly, $f_{\mathcal{X}, \mathcal{Y}}(u, v) = \frac{4}{3}$ on this region.



- (a) $f_{\mathcal{X}}(u)$ is the area of the cross-section of the pdf surface along the line u . There are two cases to be considered, as shown in the left-hand figure above. It is obvious almost by inspection that $f_{\mathcal{X}}(u) = \begin{cases} \frac{2}{3}, & 0 \leq u \leq \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$

$$(b) \mathbb{E}[\mathcal{X}] = \int_0^{1/2} u \cdot \frac{2}{3} du + \int_{1/2}^1 u \cdot \frac{4}{3} du = \frac{u^2}{3} \Big|_0^{1/2} + \frac{2u^2}{3} \Big|_{1/2}^1 = \frac{7}{12}.$$

$$\mathbb{E}[\mathcal{X}^2] = \int_0^{1/2} u^2 \cdot \frac{2}{3} du + \int_{1/2}^1 u^2 \cdot \frac{4}{3} du = \frac{2u^3}{9} \Big|_0^{1/2} + \frac{4u^3}{9} \Big|_{1/2}^1 = \frac{5}{12} \text{ and hence}$$

$$\text{var}(\mathcal{X}) = \mathbb{E}[\mathcal{X}^2] - (\mathbb{E}[\mathcal{X}])^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}.$$

- (c) By symmetry, $f_{\mathcal{X}}$ and $f_{\mathcal{Y}}$ are the same function: $f_{\mathcal{Y}}(v) = \begin{cases} \frac{2}{3}, & 0 \leq v \leq \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < v \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$ and therefore $\mathcal{Y}$ has the same mean and variance as $\mathcal{X}$.

- (d) From the symmetry of the problem, we can write

$$\mathbb{E}[\mathcal{X}\mathcal{Y}] = 2 \int_{u=1/2}^1 \int_{v=0}^u u \cdot v \cdot \frac{4}{3} dv du = 2 \int_{1/2}^1 \frac{2u^3}{3} du = \frac{u^4}{3} \Big|_{1/2}^1 = \frac{5}{16} \text{ and hence}$$

$$\text{cov}(\mathcal{X}, \mathcal{Y}) = \mathbb{E}[\mathcal{X}\mathcal{Y}] - \mathbb{E}[\mathcal{X}]\mathbb{E}[\mathcal{Y}] = \frac{5}{16} - \left(\frac{7}{12}\right)^2 = -\frac{1}{36}. \text{ Alternatively, we have}$$

$$\mathbb{E}[\mathcal{X}\mathcal{Y}] = \int_0^1 \int_0^1 u \cdot v \cdot \frac{4}{3} dv du - \int_0^{1/2} \int_0^{1/2} u \cdot v \cdot \frac{4}{3} dv du = \frac{1}{3} - \frac{1}{48} = \frac{5}{16} \text{ as before } \dots$$

- (e) For $0 < \alpha < 1$, $F_{\mathcal{Z}}(\alpha) = P\{\mathcal{Z} \leq \alpha\} = P\{\mathcal{Y} \leq \alpha\mathcal{X}\} = P\{(\mathcal{X}, \mathcal{Y}) \text{ lies below line } v = \alpha u\}$
 $= \frac{4}{3} \times \left[\frac{1}{2} \times 1 \times \alpha - \frac{1}{2} \times \frac{1}{2} \times \frac{\alpha}{2} \right] = \frac{\alpha}{2}$. On the other hand, for $1 < \alpha < \infty$,
 $1 - F_{\mathcal{Z}}(\alpha) = P\{\mathcal{Z} > \alpha\} = P\{\mathcal{Y} > \alpha\mathcal{X}\} = P\{(\mathcal{X}, \mathcal{Y}) \text{ lies above line } v = \alpha u\}$
 $= \frac{4}{3} \times \left[\frac{1}{2} \times 1 \times \alpha^{-1} - \frac{1}{2} \times \frac{1}{2} \times \frac{\alpha^{-1}}{2} \right] = \frac{1}{2\alpha}.$

Hence, $f_Z(\alpha) = \frac{d}{d\alpha} F_Z(\alpha) = \begin{cases} \frac{1}{2}, & 0 \leq \alpha \leq 1, \\ \frac{1}{2\alpha^2}, & 1 < \alpha \leq \infty, \\ 0, & \text{elsewhere.} \end{cases}$ Note that the pdf of $Z - 1$ is

$$f_Z(\alpha+1) = \begin{cases} \frac{1}{2}, & -1 \leq \alpha \leq 0, \\ \frac{1}{2(\alpha+1)^2}, & 0 < \alpha \leq \infty, \\ 0, & \text{elsewhere.} \end{cases} \quad \text{which is the pdf found in Problem 1 of Problem}$$

Set 10 (for the case $I_0 = 1$).

- (f) Given that $\mathcal{Y} = \alpha$, the conditional pdf $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha)$ is the cross-section of the pdf surface at $v = \alpha$ unitized to have area 1.

Thus, for $0 < \alpha < 1/2$, $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) \sim U[1/2, 1]$, while if $1/2 < \alpha < 1$, $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) \sim U[0, 1]$.

(g) $f_{\mathcal{X}}(u) = \int_0^1 f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) f_{\mathcal{Y}}(\alpha) d\alpha.$

- Let u denote a fixed number such that $0 < u < 1/2$. Then, for α such that $0 < \alpha < 1/2$, we have that $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) = 0$ since the conditional pdf of $\mathcal{X}$ is $U[1/2, 1]$. On the other hand, if $1/2 < \alpha < 1$, then $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) = 1$ since the conditional pdf of $\mathcal{X}$ is $U[0, 1]$. Also, $f_{\mathcal{Y}}(\alpha) = \frac{4}{3}$ if $1/2 < \alpha < 1$. Hence, for $u, 0 < u < 1/2$,

$$f_{\mathcal{X}}(u) = \int_0^1 f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) f_{\mathcal{Y}}(\alpha) d\alpha = \int_0^{1/2} 0 d\alpha + \int_{1/2}^1 1 \times \frac{4}{3} d\alpha = \frac{2}{3}.$$

- Let u denote a fixed number such that $1/2 < u < 1$. Then, for α such that $0 < \alpha < 1/2$, we have that $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) = 2$ since the conditional pdf of $\mathcal{X}$ is $U[1/2, 1]$. Note that $f_{\mathcal{Y}}(\alpha) = \frac{2}{3}$. On the other hand, if $1/2 < \alpha < 1$, then $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) = 1$ since the conditional pdf of $\mathcal{X}$ is $U[0, 1]$. Also, $f_{\mathcal{Y}}(\alpha) = \frac{4}{3}$ if $1/2 < \alpha < 1$. Hence, for

$$u, 1/2 < u < 1, f_{\mathcal{X}}(u) = \int_0^1 f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) f_{\mathcal{Y}}(\alpha) d\alpha = \int_0^{1/2} 2 \times \frac{2}{3} d\alpha + \int_{1/2}^1 1 \times \frac{4}{3} d\alpha = \frac{4}{3}.$$

This is the same answer as in part (a).