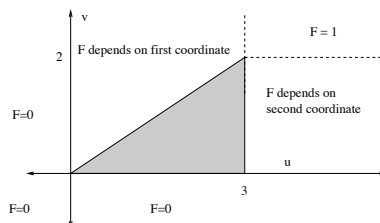


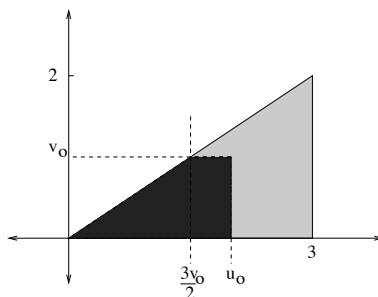
1. The CDF for a uniform distribution within a triangle

(a) The triangle has area three, so $f_{XY}(u, v) = \begin{cases} \frac{1}{3} & \text{if } 0 \leq u \leq 3 \text{ and } 3v \leq 2u \\ 0 & \text{else} \end{cases}$

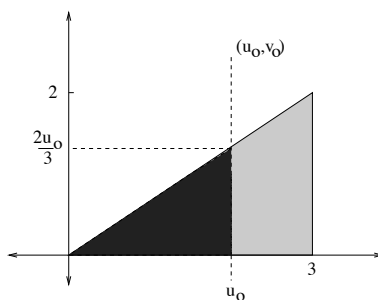
(b)



(c) If (u_o, v_o) is in the triangular region, then $F_{X,V}(u_o, v_o)$ is equal to one third the area of the darkly shaded region shown below, which is equal to $\frac{1}{3}(u_o v_o - \frac{3v_o^2}{4}) = (\frac{u_o}{3} - \frac{v_o}{4})v_o$.



If (u_o, v_o) lies above the triangular region, then $F_{X,V}(u_o, v_o)$ is equal to one third the area of the darkly shaded region shown below, or $\frac{1}{3}u_o \frac{2u_o}{3}/2$, or $\frac{u_o^2}{9}$.



Filling in these and other cases, yields:

$$F_{XY}(u_o, v_o) = \begin{cases} 0 & \text{if } u_o \leq 0 \text{ or } v_o \leq 0 \\ (\frac{u_o}{3} - \frac{v_o}{4})v_o & \text{if } 0 \leq u_o \leq 3 \text{ and } 3v_o \leq 2u_o \\ (1 - \frac{v_o}{4})v_o & \text{if } u_o \geq 3 \text{ and } 0 \leq v_o \leq 2 \\ \frac{u_o^2}{9} & \text{if } 0 \leq u_o \leq 3 \text{ and } 3v_o \geq 2u_o \\ 1 & \text{if } u_o \geq 3 \text{ and } v_o \geq 2 \end{cases}$$

Note: One can find the value in the third line by taking $u_o = 3$ in the second line.

2. A joint pmf with geometric distribution marginals

(a) Consider the sequence of phase two tests. Each wafer exposed to phase two testing passes the test with probability $p' = \frac{p_2}{p_1 + p_2}$. Thus, Y is simply the number of successes for independent trials with success probability p' . Hence, Y has the geometric distribution with parameter p' .

(b) If $1 \leq i \leq n$, then $(X, Y) = (n, i)$ if the disposition of the first n wafers is as follows: $n - i$ of the first $n - 1$ wafers fail phase one, $i - 1$ of the first $n - 1$ wafers pass phase one but not phase two, and the n^{th} wafer passes both phases. Any particular sequence of test results for the first n wafers meeting these conditions has probability $p_0^{n-i} p_1^{i-1} p_2$. Since there are $\binom{n-1}{i-1}$ possibilities for which of the first $n - 1$ wafers pass phase one, we obtain:

$$p_{X,Y}(n, i) = \begin{cases} \binom{n-1}{i-1} p_0^{n-i} p_1^{i-1} p_2 & \text{for } 1 \leq i \leq n \\ 0 & \text{else.} \end{cases}$$

(c) Given i fixed with $i \geq 1$,

$$\begin{aligned} p_Y(i) &= \sum_{n=i}^{\infty} \binom{n-1}{i-1} p_0^{n-i} p_1^{i-1} p_2 \\ &= \left[\sum_{n=i}^{\infty} \binom{n-1}{i-1} p_0^{n-i} (1 - p_0)^i \right] \left(\frac{p_1}{1 - p_0} \right)^{i-1} \frac{p_2}{1 - p_0} \\ &= (1 - p')^{i-1} p', \end{aligned}$$

where we used the fact that the quantity in the square brackets, being the sum over the negative binomial distribution with parameter p_0 and i , is one.

3. Uniform distribution on a rotated square

(a) X and Y are not independent because, for example, the footprint of their joint pdf is not rectangular. The area of the square region is 2 so that the pdf of (X, Y) is $\frac{1}{2}$ within the region. Integrating out the v variable of the joint density function yields the triangle shaped pdf:

$$f_X(u) = \begin{cases} \int_{-1-u}^{1+u} \frac{1}{2} dv = 1 + u, & -1 < u \leq 0 \\ \int_{u-1}^{1-u} \frac{1}{2} dv = 1 - u, & 0 < u \leq 1 \\ 0 & \text{else} \end{cases}$$

Equivalently,

$$f_X(u) = \begin{cases} 1 - |u| & -1 \leq u \leq 1 \\ 0 & \text{else} \end{cases}$$

Since the joint pdf is symmetric in u and v , it follows that f_Y is the same as f_X .

(b)

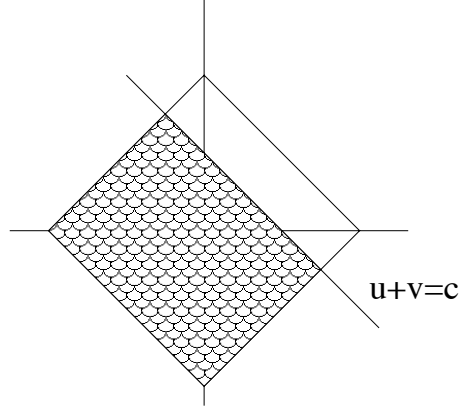
$$E[X] = \int_{-\infty}^{\infty} u f_X(u) du = \int_{-1}^1 u(1 - |u|) du = 0$$

because an odd function integrates to zero.

$$E[X^2] = \int_{-\infty}^{\infty} u f_X(u) du = \int_{-1}^1 u^2(1 - |u|) du = 2 \int_0^1 u^2(1 - u) du = 2 \int_0^1 u^2 - u^3 du = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{6}$$

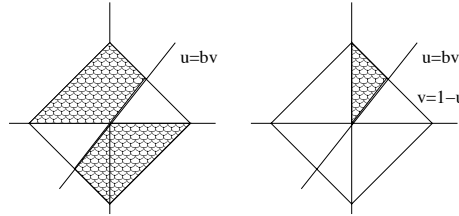
So $\text{Var}(X) = E[X^2] - E[X]^2 = \frac{1}{6}$.

(c) The range of A is the interval $[-1, 1]$. If c is in this interval, then the event $\{A \leq c\}$ is the same as (X, Y) falling within the halfspace $\{(u, v) : u + v \leq c\}$. The halfspace intersects the rotated square in a rectangle of dimensions $\frac{1+c}{\sqrt{2}} \times \sqrt{2}$. See the figure.



The area of this rectangle is $1 + c$. Since the pdf of (X, Y) is $\frac{1}{2}$ within the rectangle, $P(A \leq c) = \frac{1+c}{2}$. Hence, A is uniformly distributed over the interval $[-1, 1]$.

(d) The range of C is the whole real line. Our approach will be to first find the CDF and pdf of C within the region $b \geq 0$. So let $b \geq 0$. Then $F_C(b)$ is one half the area of the shaded region in the diagram on the left below, which is 0.5 plus the area of the shaded region in the diagram on the right.



Take the base of the triangle on the right to be the unit length segment from $(0, 0)$ to $(0, 1)$. Then the height is the value of u at which the line $\{v = bv\}$ intersects the edge of the square, given by the equation $u + v = 1$. Solving for u yields $u = \frac{b}{1+b}$. The area of the shaded triangle is thus $\frac{u}{2} = \frac{b}{2(1+b)}$ so that $F_C(b) = 0.5 + \frac{b}{2(1+b)}$ for $b \geq 0$. Differentiating this with respect to b yields that $f_C(b) = \frac{1}{2(1+b)^2}$ for $b \geq 0$.

We could find the pdf for negative b by a similar method. However, it is less work to notice that (X, Y) and $(-X, Y)$ have the same joint distribution, so that C has the same distribution as $-C$. Thus, f_C is an even function: $f_C(-b) = f_C(b)$. Therefore, $f_C(b) = \frac{1}{2(1+|b|)^2}$ for all real values of b .

4. Uniform distribution over a parallelogram

(a) area=base $\times$ height=1, so density inside region is 1

(b)

$$f_X(u) = \begin{cases} \frac{x}{2} & : 0 \leq u \leq 1 \\ \frac{1}{2} & : 1 \leq u \leq 2 \\ \frac{3-x}{2} & : 2 \leq u \leq 3 \\ 0 & : \text{else} \end{cases}$$

(c) $E[X] = \frac{3}{2}$, $\text{Var}(X) = \frac{5}{12}$

(d) For $0 \leq a \leq 1$:

$$f_{Y|X}(v|a) = \begin{cases} \frac{2}{a} & : 0 \leq v \leq \frac{a}{2} \\ 0 & : \text{else} \end{cases}$$

(e) For $1 \leq a \leq 2$:

$$f_{Y|X}(v|a) = \begin{cases} 2 & : \frac{a-1}{2} \leq v \leq \frac{a}{2} \\ 0 & : \text{else} \end{cases}$$

(f)

$$E[Y|X=a] = \begin{cases} \frac{a}{4} & : 0 \leq a \leq 1 \\ \frac{2a-1}{4} & : 1 \leq a \leq 2 \\ \frac{a+1}{4} & : 2 \leq a \leq 3 \\ 0 & : \text{else} \end{cases}$$

(So the slope is 1/4 over the intervals [0,1] and [2,3], and the slope is 1/2 over the interval [1,2].)

5. Functions of independent exponential random variables

(a) Z takes values in the positive real line, so $F_Z(c) = 0$ for $c < 0$. For $c \geq 0$,

$$\begin{aligned} F_Z(c) &= P(\min\{X_1, X_2\} \leq c) = P(X_1 \leq c \text{ or } X_2 \leq c) \\ &= 1 - P(X_1 > c \text{ and } X_2 > c) \\ &= 1 - P(X_1 > c)P(X_2 > c) = 1 - e^{-\lambda_1 c}e^{-\lambda_2 c} = 1 - e^{-(\lambda_1 + \lambda_2)c} \end{aligned}$$

Differentiating yields:

$$f_Z(c) = \begin{cases} (\lambda_1 + \lambda_2)e^{-(\lambda_1 + \lambda_2)c} & \text{for } c \geq 0 \\ 0 & \text{else} \end{cases}$$

That is, Z has the exponential distribution with parameter $\lambda_1 + \lambda_2$.

(b) R takes values in the positive real line so that $F_R(a) = 0$ for $a < 0$. So consider $a > 0$. By independence the joint pdf of X_1 and X_2 is the product of the individual densities. So for $a \geq 0$,

$$\begin{aligned} F_R(a) &= P\left(\frac{X_1}{X_2} \leq a\right) = P(X_1 \leq aX_2) = \int_0^\infty \int_0^{av} \lambda_1 e^{-\lambda_1 u} \lambda_2 e^{-\lambda_2 v} du dv \\ &= \int_0^\infty (1 - e^{-a\lambda_1 v}) \lambda_2 e^{-\lambda_2 v} dv = 1 - \frac{\lambda_2}{a\lambda_1 + \lambda_2}. \end{aligned}$$

Differentiating yields:

$$f_R(a) = \begin{cases} \frac{\lambda_1 \lambda_2}{(\lambda_1 a + \lambda_2)^2} & a \geq 0 \\ 0 & \text{else} \end{cases}$$