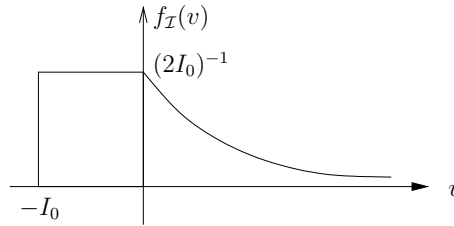


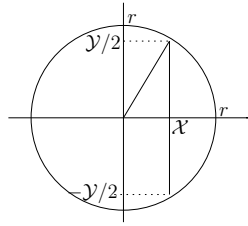
1. (a) $\mathcal{I}$ can take on values in the range $(-I_0, \infty)$.
 (b) $F_{\mathcal{I}}(v) = 0$ for $v < -I_0$. For any $v > -I_0$, $F_{\mathcal{I}}(v)$
 $= P\{\mathcal{I} \leq v\} = P\{I_0(\exp(\mathcal{V}) - 1) \leq v\} = P\left\{\mathcal{V} \leq \ln\left(1 + \frac{v}{I_0}\right)\right\} = F_{\mathcal{V}}\left(\ln\left(1 + \frac{v}{I_0}\right)\right)$.
 (c) For $v > -I_0$, $f_{\mathcal{I}}(v)$

$$= f_{\mathcal{V}}(\ln(1 + v/I_0)) \frac{1}{1 + v/I_0} \times \frac{1}{I_0} = \frac{f_{\mathcal{V}}(\ln(1 + v/I_0))}{v + I_0} = \begin{cases} \frac{I_0/2}{(v + I_0)^2}, & v \geq 0, \\ \frac{1}{2I_0}, & -I_0 < v < 0. \end{cases}$$

Note that the pdf has constant value $1/(2I_0)$ from $v = -I_0$ to $v = 0$.



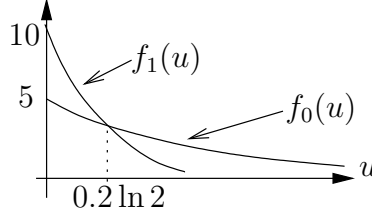
2. (a) $\mathcal{Y}$ takes on values in $[0, 1]$ and hence $F_{\mathcal{Y}}(v) = 0$ for $v < 0$, and $F_{\mathcal{Y}}(v) = 1$ for $v > 1$.
 For $0 \leq v \leq 1$, $F_{\mathcal{Y}}(v) = P\{\mathcal{Y} \leq v\} = P\{\mathcal{X}^2 \leq v\} = P\{-\sqrt{v} \leq \mathcal{X} \leq \sqrt{v}\} = \sqrt{v}$.
 Hence $f_{\mathcal{Y}}(v) = \frac{1}{2\sqrt{v}}$ if $0 \leq v \leq 1$, and $f_{\mathcal{Y}}(v) = 0$, otherwise.
 (b) $\mathcal{Z}$ takes on values in $[-1, 1]$ and hence, $F_{\mathcal{Z}}(v) = 0$ for $v < -1$, and $F_{\mathcal{Z}}(v) = 1$ for $v > 1$.
 For $0 \leq v \leq 1$, $F_{\mathcal{Z}}(v) = P\{\mathcal{Z} \leq v\} = P\{g(\mathcal{X}) \leq v\} = P\{\mathcal{X} \leq \sqrt{v}\} = \frac{1}{2}[1 + \sqrt{v}]$.
 For $-1 \leq v \leq 0$, $F_{\mathcal{Z}}(v) = P\{\mathcal{Z} \leq v\} = P\{g(\mathcal{X}) \leq v\} = P\{\mathcal{X} \leq -\sqrt{-v}\} = \frac{1}{2}[1 - \sqrt{-v}]$.
 Hence, $f_{\mathcal{Z}}(v) = \frac{1}{4\sqrt{|v|}}$ if $0 \leq |v| \leq 1$, and $f_{\mathcal{Z}}(v) = 0$, otherwise. Note that the pdf is an even function, and approaches $+\infty$ as v approaches 0 from either side.
3. Without loss of generality, $\mathcal{X}$ is a random point in the interval $[0, r]$ on the horizontal axis, and the chord is parallel to the vertical axis as shown in the diagram.



Note that $\mathcal{Y}$ takes on values in $[0, 2r]$ and hence $F_{\mathcal{Y}}(v) = 0$ for $v < 0$, and $F_{\mathcal{Y}}(v) = 1$ for $v > 2r$. Furthermore, $\mathcal{X}^2 + (\mathcal{Y}/2)^2 = r^2$, and hence for any v , $0 \leq v \leq 2r$,
 $F_{\mathcal{Y}}(v) = P\{\mathcal{Y} \leq v\} = P\{2\sqrt{r^2 - \mathcal{X}^2} \leq v\} = P\{\mathcal{X}^2 \geq r^2 - v^2/4\} = P\{\mathcal{X} \geq \sqrt{r^2 - v^2/4}\}$
 $= r^{-1}(r - \sqrt{r^2 - v^2/4})$ since $\mathcal{X}$ is uniformly distributed on $[0, r]$. It follows that
 $f_{\mathcal{Y}}(v) = \frac{d}{dv}F_{\mathcal{Y}}(v) = \frac{v}{2r\sqrt{4r^2 - v^2}}$ for $0 \leq v \leq 2r$, and $f_{\mathcal{Y}}(v) = 0$ otherwise.

4. (a) The pmf of $\mathcal{Y}$ is $p_{\mathcal{Y}}(\alpha) = p_{\mathcal{Y}}(-\alpha) = \frac{1}{2}$.
- (b) $E[\mathcal{Z}] = \int_0^\infty (u-\alpha)^2 \phi(u) du + \int_{-\infty}^0 (u+\alpha)^2 \phi(u) du = \int_{-\infty}^\infty (u^2 + \alpha^2) \phi(u) du - 4 \int_0^\infty \alpha u \phi(u) du$
 $= 1 + \alpha^2 - 2\sqrt{\frac{2}{\pi}}\alpha$ where we have used the fact (cf. handout notes) that $u\phi(u)$ has a known antiderivative in arriving at the result. Now, a quadratic function $ax^2 + 2bx + c$ has extremum $c - \frac{b^2}{a}$ at $x = -\frac{b}{a}$, the average of the roots, and this extremum is a minimum if $a > 0$, maximum if $a < 0$. So we see $E[\mathcal{Z}]$ has minimum value $1 - \frac{2}{\pi}$ at $\alpha = \sqrt{\frac{2}{\pi}}$.
- (c) $p_{\mathcal{W}}(3) = p_{\mathcal{W}}(-3) = \Phi(-2.5) = 0.0062$. $p_{\mathcal{W}}(2) = p_{\mathcal{W}}(-2) = \Phi(2.5) - \Phi(1.5) = 0.0606$.
 $p_{\mathcal{W}}(1) = p_{\mathcal{W}}(-1) = \Phi(1.5) - \Phi(0.5) = 0.2417$. $p_{\mathcal{W}}(0) = \Phi(0.5) - \Phi(-0.5) = 0.3830$.
- (d) $\mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_0$ are *Bernoulli* random variables with parameters $p_2 = P\{\mathcal{W} < 0\} = 0.3085$,
 $p_1 = P\{\mathcal{W} \in \{-2, -1, 2, 3\}\} = 0.3691$, and $p_0 = P\{\mathcal{W} \in \{-3, -1, 1, 3\}\} = 0.4958$ respectively.
5. (a) It is easy to see that $\{Y > T\} = \{\mathcal{X}_1 > T\} \cup \{\mathcal{X}_2 > T\}$ and hence, by independence,
- $$\begin{aligned} P\{Y > T\} &= P\{\mathcal{X}_1 > T\} + P\{\mathcal{X}_2 > T\} - P\{\{\mathcal{X}_1 > T\} \cap \{\mathcal{X}_2 > T\}\} \\ &= P\{\mathcal{X}_1 > T\} + P\{\mathcal{X}_2 > T\} - P\{\mathcal{X}_1 > T\}P\{\mathcal{X}_2 > T\} \end{aligned}$$
- (b) Since $P\{\mathcal{X}_i > T\} = \exp(-\lambda T)$, we get that
- $$P\{\mathcal{Y} > T\} = \exp(-\lambda T) + \exp(-\lambda T) - (\exp(-\lambda T))^2 = 2\exp(-\lambda T) - \exp(-2\lambda T).$$
- i. $E[\mathcal{Y}] = \int_0^\infty P\{\mathcal{Y} > T\} dT = \int_0^\infty 2\exp(-\lambda T) - \exp(-2\lambda T) dT = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2}\lambda^{-1}$.
- ii. Let $p = \exp(-\lambda T)$. Then, $P\{\mathcal{Y} > T\} = 2p - p^2 = \frac{1}{2}$, which has only one solution $p = 1 - 1/\sqrt{2}$ in the range $0 \leq p \leq 1$. Thus, the median value of $\mathcal{Y}$ is the solution T to $\exp(-\lambda T) = 1 - 1/\sqrt{2}$, giving $T = \lambda^{-1} \ln(2 + \sqrt{2})$.
- iii. $f_{\mathcal{Y}}(T) = -\frac{d}{dT}P\{Y > T\} = 2\lambda \exp(-\lambda T)(1 - \exp(-\lambda T))$ for $T > 0$. Hence,
- $$h_{\mathcal{Y}}(T) = \frac{f_{\mathcal{Y}}(T)}{P\{Y > T\}} = \frac{2\lambda(1 - \exp(-\lambda T))}{2 - \exp(-\lambda T)} = 2\lambda \left[1 - \frac{1}{2 - \exp(-\lambda T)} \right]$$
- which has value 0 at $T = 0$ and slowly increases to λ , the hazard rate of the single-component system, as $T \rightarrow \infty$.
- (c) The dual-component system has a longer MTBF ($1.5\lambda^{-1}$ compared to λ^{-1}), longer half-life ($\lambda^{-1} \ln(2 + \sqrt{2})$ compared to $\lambda^{-1} \ln(2)$), and smaller hazard rate than the single component system.

6. (a) The pdfs are as shown below.



- (b) $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{10 \cdot \exp(-10u)}{5 \cdot \exp(-5u)} = 2 \cdot \exp(-5u)$
 which has value 2 at $u = 0$ and decays away to 0 as $u \rightarrow \infty$. Note that $\Lambda(u) > 1$ for $u < 0.2 \ln 2$. Thus, the likelihood ratio test is equivalent to deciding in favor of H_1 if the observed value of $\mathcal{X}$ is *smaller* than the threshold $0.2 \ln 2$. Equivalently, $\Gamma_1 = (0, 0.2 \ln 2)$, $\Gamma_0 = (0.2 \ln 2, \infty)$.

$$(c) \quad P_{FA} = \int_{\Gamma_1} f_0(u) du = \int_0^{0.2 \ln 2} 5 \cdot \exp(-5u) du = -\exp(-5u) \Big|_0^{0.2 \ln 2} = -\frac{1}{2} - (-1) = \frac{1}{2}.$$

$$P_{MD} = \int_{\Gamma_0} f_1(u) du = \int_{0.2 \ln 2}^{\infty} 10 \cdot \exp(-10u) du = -\exp(-10u) \Big|_{0.2 \ln 2}^{\infty}$$

$$= 0 - (-\exp(-2 \ln 2)) = \frac{1}{4}.$$

- (d) $\Lambda(u) = 2 \cdot \exp(-5u) > \frac{\pi_0}{\pi_1}$ for $u < 0.2 \ln \left(\frac{2\pi_1}{\pi_0} \right) = 0.2 \ln 2 + 0.2 \ln \left(\frac{\pi_1}{\pi_0} \right) = \xi$. Thus, the minimum-error-probability decision rule is equivalent to deciding in favor of H_1 if the observed value of $\mathcal{X}$ is smaller than ξ . Note that $\xi < 0$ if $\pi_0 > 2\pi_1$, that is, if $\pi_0 > 2/3$.

- (e) If $\pi_0 = 1/3$, then $\xi = 0.2 \ln 4$. Hence,

$$P_{FA} = \int_0^{0.2 \ln 4} 5 \cdot \exp(-5u) du = -\exp(-5u) \Big|_0^{0.2 \ln 4} = -\frac{1}{4} - (-1) = \frac{3}{4}.$$

$$P_{MD} = \int_{0.2 \ln 4}^{\infty} 10 \cdot \exp(-10u) du = -\exp(-10u) \Big|_{\xi}^{\infty}$$

$$= 0 - (-\exp(-2 \ln 4)) = \frac{1}{16}.$$

The average error probability thus is $\bar{P}_e = \frac{1}{3}P_{FA} + \frac{2}{3}P_{MD} = \frac{7}{24}$. Note that since $\pi_0 < \pi_1$, the Bayesian decision rule allows P_{FA} to increase in return for a decrease in P_{MD} because the latter is weighted more heavily.

- (f) If the decision rule always decides H_1 is the true hypothesis it makes errors if and only if H_0 is the true hypothesis. Hence, $\bar{P}_e = \pi_0$.

- (g) When $\pi_0 > 2/3$, the threshold ξ is less than 0. Since $\mathcal{X}$ takes on nonnegative values, it is always larger than the threshold, and hence the decision is always H_0 . The average error probability is π_1 , and since this is the minimum-error-probability rule, we cannot do any better than this. Note that $\pi_1 < 1/3$.

When $\pi_0 > 2/3$, it follows that $\pi_0 > 2\pi_1$. The average probability of error for the maximum-likelihood rule is $\pi_0 \cdot (1/2) + \pi_1 \cdot (1/4) > 2\pi_1 \cdot (1/2) + \pi_1 \cdot (1/3) = 1.25\pi_1$.