

1. Moments of the exponential distribution

$$(a) E[X^n] = \int_0^\infty x^n \mu e^{-\mu x} dx = -nx^{n-1}e^{-\mu x}|_0^\infty + \int_0^\infty x^{n-1} ne^{-\mu x} dx = 0 + \frac{nE[X^{n-1}]}{\mu}.$$

$$(b) E[X^n] = \frac{nE[X^{n-1}]}{\mu} = \frac{n(n-1)E[X^{n-2}]}{\mu^2} = \dots = \frac{n!}{\mu^n} E[X^0] = \frac{n!}{\mu^n}$$

$$(c) E[e^{\alpha X}] = \int_0^\infty e^{\alpha x} \mu e^{-\mu x} dx = \mu \int_0^\infty e^{-(\mu-\alpha)x} dx = -\frac{\mu}{\mu-\alpha} e^{-(\mu-\alpha)x}|_0^\infty = \frac{\mu}{\mu-\alpha}$$

$$(d) \text{ On one hand, } E[e^{\alpha X}] = E\left[\sum_{n=0}^\infty \frac{\alpha^n X^n}{n!}\right] = \sum_{n=0}^\infty \frac{\alpha^n E[X^n]}{n!}.$$

On the other hand, $\frac{\mu}{\mu-\alpha} = \frac{1}{1-\frac{\alpha}{\mu}} = \sum_{n=0}^\infty \frac{\alpha^n}{\mu^n}$. These two series are equal by part (c), so the coefficients of α^n must be equal for all n . Therefore, $\frac{E[X^n]}{n!} = \frac{1}{\mu^n}$, or $E[X^n] = \frac{n!}{\mu^n}$.

2. Splitting property of the Poisson distribution

(a) Given there are n flips, N_1 has the binomial distribution with parameters n and p . Thus,

$$P(N_1 = n_1 | N = n) = \binom{n}{n_1} p^{n_1} (1-p)^{n-n_1}.$$

(b)

$$\begin{aligned} P(N_1 = n_1, N_2 = n_2) &= P(N_1 = n_1 | N = n_1 + n_2) p_N(n_1 + n_2) \\ &= \binom{n_1 + n_2}{n_1} p^{n_1} (1-p)^{n_2} \frac{e^{-\mu} \mu^{n_1+n_2}}{(n_1 + n_2)!} \end{aligned}$$

(c) Which can be written as

$$\begin{aligned} P(N_1 = n_1, N_2 = n_2) &= e^{-\mu} \frac{(\mu p)^{n_1}}{n_1!} \frac{(\mu(1-p))^{n_2}}{n_2!} \\ &= p_{N_1}(n_1) p_{N_2}(n_2) \end{aligned}$$

where $p_{N_1}(n_1) = \frac{(\mu p)^{n_1} e^{-\mu p}}{n_1!}$ and $p_{N_2}(n_2) = \frac{(\mu(1-p))^{n_2} e^{-\mu(1-p)}}{n_2!}$. That is, N_1 has the Poisson distribution with parameter μp , and N_2 has the Poisson distribution with parameter $\mu(1-p)$.

(Remark: Here is another explanation of the fact that N_1 is a Poisson random variable with parameter μp . Based on the way the Poisson distribution arises in practice, we can imagine that N is the number of successes for n independent trials in an experiment with success probability p' , where n is very large, p' is very small, and $np' = \mu$. For each success, we flip a coin with bias p , and if heads shows we say the trial is a super success. The number of super successes, N_1 , is thus a binomial random variable with parameters n and $p'p$, where n is very large, $p'p$ is very small (because p' is small) and $np'p = \mu p$. Thus, N_1 is a Poisson random variable.)

3. Arrival times for a Poisson process

$$(a) P(N(t) = i) = \frac{e^{-2t} (2t)^i}{i!} \text{ for nonnegative integers } i \text{ (Poisson distribution, mean } 2t)$$

$$(b) P(E_1) = P(N(3.5) = 0) = \frac{e^{-7} (7)^0}{0!} = e^{-7} = 0.00091$$

$P(E_2) = 0.00091$ since E_2 is the same event as E_1

$$P(E_3) = P(N(3.5) \leq 2) = P(N(3.5) = 0) + P(N(3.5) = 1) + P(N(3.5) = 2) = e^{-7} (1 + 7 + \frac{7^2}{2}) = 0.0296$$

$P(E_4) = 0.0296$ since E_4 is the same event as E_3

$$P(E_5) = P(N(t) \leq 2) = P(N(t) = 0) + P(N(t) = 1) + P(N(t) = 2) = e^{-2t} (1 + 2t + \frac{(2t)^2}{2})$$

$$P(E_6) = 1 - P(E_5) = 1 - e^{-2t} (1 + 2t + \frac{(2t)^2}{2})$$

(c) Differentiating yields $f(t) = e^{-2t} (2(1 + 2t + \frac{(2t)^2}{2}) - 2 - 4t) = e^{-2t} \frac{(2t)^2}{2}$ which indeed is the gamma density with parameters $(i = 3, \mu = 2)$.

(d) The expected time between arrivals is $1/\mu$, so the expected time until the tenth arrival is $10/\mu = 5$ minutes.

4. Poisson process probabilities

(a) The numbers of arrivals in the disjoint intervals are independent, Poisson random variables with mean μ . Thus, the probability is $(\mu e^{-\mu})^3 = \mu^3 e^{-3\mu}$.

(b) We wish to determine the probability of the event $A = \{N(0, 2] = 2, N(1, 3] = 2\}$. A is the union of the disjoint events B_{020} , B_{111} , and B_{202} , where $B_{ijk} = \{N(0, 1] = i, N(1, 2] = j, N(2, 3] = k\}$. So $P(A) = P(B_{020}) + P(B_{111}) + P(B_{202}) = e^{-\mu}(\frac{\mu^2}{2}e^{-\mu})e^{-\mu} + (\mu e^{-\mu})^3 + (\frac{\mu^2}{2}e^{-\mu})e^{-\mu}(\frac{\mu^2}{2}e^{-\mu}) = (\frac{\mu^2}{2} + \mu^3 + \frac{\mu^4}{4})e^{-3\mu}$.

(c) $P(B_{020}|A) = \frac{P(B_{020} \cap A)}{P(A)} = \frac{P(B_{020})}{P(A)} = \frac{\mu^2/2}{(\mu^2/2 + \mu^3 + \mu^4/4)} = 2/(2 + 4\mu + \mu^2)$. (We have applied Bayes rule, which in this case is the definition of the conditional probability $P(B_{020}|A)$ and use of the law of total probability for finding $P(A)$.)

5. Calculating probabilities of events involving a Gaussian random variable

In parts (a)-(c) we use the fact that $\frac{X-10}{4}$ is a $N(0,1)$ random variable.

(a) $P(X \geq 15) = P(\frac{X-10}{4} \geq \frac{15-10}{4}) = Q(\frac{15-10}{4}) = Q(1.25) = 1 - \Phi(1.25) = 0.1056$.

(b) $P(X \leq 5) = P(\frac{X-10}{4} \leq \frac{5-10}{4}) = P(\frac{X-10}{4} \leq -1.25) = Q(1.25) = 0.1056$.

(c) $P(X^2 \geq 400) = P(X \geq 20) + P(X \leq -20) = P(\frac{X-10}{4} \geq 2.5) + P(\frac{X-10}{4} \leq -7.5) = Q(2.5) + Q(7.5) \approx Q(2.5) = 1 - \Phi(2.5) = 0.0062$ (Note: $Q(7.2) < 10^{-12}$.)

(d) $P(X = 2) = 0$ because X is a continuous type random variable.

6. Normal approximation to the binomial distribution

Do problem 20, p. 230 of *Ross* using the binomial approximation. The number of people S favoring the proposition is a binomial random variable with parameters $(100, 0.65)$. It has mean $\mu = 65$ and variance $\sigma^2 = 100(0.65)(1 - 0.65) = 22.75$. The DeMoivre-Laplace limit theorem tells us that S is well approximated by a normal random variable with parameters μ and σ^2 .

(a) $P(S \geq 50) = P(S \geq 49.5) = P(\frac{S-65}{\sqrt{22.75}} \geq \frac{49.5-65}{\sqrt{22.75}}) \approx Q(-3.25) = \Phi(3.25) = 0.9994$

(b) $P(60 \leq S \leq 70) = P(59.5 \leq S \leq 70.5) = P(\frac{59.5-65}{\sqrt{22.75}} \leq \frac{S-65}{\sqrt{22.75}} \leq \frac{70.5-65}{\sqrt{22.75}}) \approx \Phi(1.153) - \Phi(-1.153) = 2\Phi(1.153) - 1 \approx 0.751$

(c) $P(S < 75) = P(S \leq 74.5) = P(\frac{S-65}{\sqrt{22.75}} \leq \frac{74.5-65}{\sqrt{22.75}}) \approx \Phi(2.003) \approx 0.9774$