

1. (a) We break the integral into a sum of integrals over the intervals $[k, k+1)$ to get

$$E[\mathcal{X}] = \int_0^\infty P\{\mathcal{X} > u\} du = \sum_{k=0}^\infty \int_k^{k+1} P\{\mathcal{X} > u\} du.$$

But, for each $u \in [k, k+1)$, $P\{\mathcal{X} > u\}$ has the same value $P\{\mathcal{X} > k\}$. The k -th integral thus has a constant integrand $P\{\mathcal{X} > k\}$ and therefore evaluates to $P\{\mathcal{X} > k\}$. Furthermore, if $i = k+1$, then $P\{\mathcal{X} > k\} = P\{X \geq k+1\} = P\{X \geq i\}$. Hence $E[\mathcal{X}] = \sum_{k=0}^\infty P\{X \geq k+1\} = \sum_{i=1}^\infty P\{X \geq i\}$.

- (b) For $k = 0, 1, 2, \dots$, $P\{\mathcal{X} > k\} = q^k$ where $q = 1 - p$. Therefore,

$$E[\mathcal{X}] = \sum_{k=0}^\infty P\{X > k\} = \sum_{k=0}^\infty q^k = \frac{1}{1-q} = \frac{1}{p}.$$

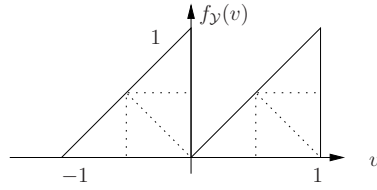
2. If $f(u)$ is a nonnegative (or nonpositive) function with finite nonzero area A , then $A^{-1} \cdot f(u)$ is a valid pdf. This does not work if $f(u)$ takes on both positive and negative values.

- (a) $f(u) = 2u$, $0 < u < 1$ is a valid pdf.
 (b) $f(u) = |u|$, $|u| < \frac{1}{2}$ is not a valid pdf, but $4 \cdot f(u)$ is.
 (c) $f(u) = 1 - |u|$, $|u| < 1$ is a valid pdf.
 (d) $f(u) = \ln u$, $0 < u < 1$ is not a valid pdf but $-f(u)$ is.
 (e) $f(u) = \ln u$, $0 < u < 2$ is not a valid pdf, nor is $C \cdot f(u)$ a valid pdf for any choice of C .
 (f) $f(u) = \frac{2}{3}(u-1)$, $0 < u < 3$, is not a valid pdf, nor is $C \cdot f(u)$ a valid pdf for any choice of C .
 (g) $f(u) = \exp(-2u)$, $u > 0$ is not a valid pdf but $2 \cdot f(u)$ is.
 (h) $f(u) = 4e^{-2u} - e^{-u}$, $u > 0$, is not a valid pdf, nor is $C \cdot f(u)$ a valid pdf for any choice of C .
 (i) $f(u) = \exp(-|u|)$, $|u| < 1$ is not a valid pdf but $e/[2(e-1)] \cdot f(u)$ is.

3. (a) $P\{\mathcal{X} > 20\} = \int_{20}^\infty \frac{10}{u^2} du = \frac{-10}{u} \Big|_{20}^\infty = \frac{1}{2}$.
 (b) $F_{\mathcal{X}}(u_0) = \int_{-\infty}^{u_0} f_{\mathcal{X}}(u) du = \int_{10}^{u_0} \frac{10}{u^2} du = \frac{-10}{u} \Big|_{10}^{u_0} = 1 - \frac{10}{u_0}$. Note that $F_{\mathcal{X}}(20) = \frac{1}{2}$; it should be!
 (c) $P\{\mathcal{X} > 15\} = 1 - F_{\mathcal{X}}(15) = \frac{10}{15} = \frac{2}{3}$. Assuming that the device failures are independent, the probability that at least 3 of the 6 last for 15 hours (i.e., at most 3 fail) is a binomial probability:

$$P\{\text{at least 3 of 6 devices work for 15 hours}\} = P\{\text{at most 3 failures}\} = \sum_{i=0}^3 \binom{6}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{6-i}.$$

4. The pdf of $\mathcal{Y}$ is as shown below.



The dotted lines marked (together with the fact that the total area under the pdf is 1) allow us to find the probabilities merely by counting small triangles, that is, without explicit use of integral calculus.

By inspection, $P\left\{|\mathcal{Y}| < \frac{1}{2}\right\} = \frac{1}{2}$ and $P\left\{\mathcal{Y} > 0 \mid \mathcal{Y} < \frac{1}{2}\right\} = \frac{P\{0 < \mathcal{Y} < \frac{1}{2}\}}{P\{\mathcal{Y} < \frac{1}{2}\}} = \frac{1}{5}$.

Finally, $E[\mathcal{Y}] = \int_{-1}^0 v(1+v) dv + \int_0^1 v^2 dv = \frac{v^2}{2} + \frac{v^3}{3} \Big|_{-1}^0 + \frac{v^3}{3} \Big|_0^1 = \frac{-1}{6} + \frac{1}{3} = \frac{1}{6}$.

Even this can be obtained without integration. As far as moments are concerned, the two triangles can be represented by masses of $\frac{1}{2}$ sitting at $\frac{2}{3}$ and $-\frac{1}{3}$ respectively, and so the center of mass is at the midpoint of the two locations: $(\frac{2}{3} - \frac{1}{3})/2 = 1/6$.

5. (a) $P\{\text{demand can be satisfied}\} = P\left\{\mathcal{X} \leq \frac{1}{2}\right\} = \int_0^{0.5} 5(1-u)^4 du = -(1-u)^5 \Big|_0^{0.5} = \frac{31}{32}$.

(b) We want the minimum value of C such that $P\{\mathcal{X} > C\} \leq 10^{-5}$. But,

$$P\{\mathcal{X} > C\} = \int_C^1 5(1-u)^4 du = -(1-u)^5 \Big|_C^1 = (1-C)^5 \leq 10^{-5} \text{ if } C \geq 0.9.$$

(c) $\mathcal{Y} = \min\{\mathcal{X}, C\} = \begin{cases} \mathcal{X}, & \text{if } \mathcal{X} \leq C, \\ C, & \text{if } \mathcal{X} > C. \end{cases}$

(d) The weekly *gross* profit is $\$640\mathcal{Y}$ and the *average* weekly gross profit is

$$E[640\mathcal{Y}] = \int_0^C 5(1-u)^4(640u) du + \int_C^1 5(1-u)^4(640C) du = \frac{640}{6} [1 - (1-C)^6].$$

As a function of C , this increases from 0 at $C = 0$ to $640/6$ when $C = 1$. The average *net* profit is $\$E[(640\mathcal{Y} - 20C)] = E[(640\mathcal{Y})] - 20C = \frac{640}{6} [1 - (1-C)^6] - 20C$ whose maximum can be found by the usual calculus method to be at $C = \frac{1}{2}$. Thus, a 500 gallon tank that costs \$10 per week to rent gives an average gross profit of $\$630/6$ and the maximum average net profit of $\$570/6$ per week. In contrast, a full-sized 1000 gallon tank gives only a slightly larger gross profit of $\$640/6$ per week, but a smaller net profit of $\$520/6$ per week. Of course, there may be intangible losses in goodwill if the tank is emptied before the week ends. However, the probability of this happening is only $\frac{1}{32}$ when $C = \frac{1}{2}$.