

1. The reliability of multiple connections

The probability of total outage on a given day is p^k , so the probability of total outage at least one day during a year is $1 - (1 - p^k)^{365}$. For $p = 0.001$, this gives values 0.3059 for $k = 1$, 0.000365 for $k = 2$, and 3.65×10^{-7} for $k = 3$. We find that $k = 2$ suffices (even for a leap year).

2. Failure rate function for discrete random variables

(a) Since $\{T = k\} \subset \{T \geq k\}$, the definitions of conditional probability and pmf yield $\lambda_T(k) = P(T = k|T \geq k) = \frac{P(T=k)}{P(T \geq k)} = \frac{p_T(k)}{\sum_{l=k}^{\infty} p_T(l)}$. Or, equivalently, $\lambda_T(k) = \frac{p_T(k)}{1 - \sum_{l=1}^{k-1} p_T(l)}$. (In particular, $\lambda_T(1) = p_T(1)$. Note that $\lambda_T(k)$ is only well defined if $P(T \geq k) > 0$.)

(b) By definition, $\lambda_T(k_f) = 1$ if and only if $P(T = k_f) = P(T \geq k_f) > 0$. So $\lambda_T(k_f) = 1$ if and only if $P(T > k_f) = P(T \geq k_f) - P(T = k_f) = 0$ and $P(T = k_f) > 0$. Thus, the condition $\lambda_T(k_f) = 1$ means that T has a largest possible value, and that value is k_f .

(c) For $1 \leq k \leq n$, $\lambda_T(k) = \frac{p_T(k)}{1 - \sum_{l=1}^{k-1} p_T(l)} = \frac{\frac{1}{n}}{1 - \frac{k-1}{n}} = \frac{1}{n-k+1}$. (That is, the first n values of λ_T are $\frac{1}{n}, \frac{1}{n-1}, \dots, \frac{1}{1}$.)

(d) One solution is to first derive a general formula. Note that

$$\begin{aligned} P(T \geq k+1) &= P(\{T \geq k+1\} \cap \{T \geq k\}) \\ &= P(T \geq k+1|T \geq k)P(T \geq k) \\ &= (1 - P(T = k|T \geq k))P(T \geq k) \\ &= (1 - \lambda_T(k))P(T \geq k) \end{aligned} \tag{1}$$

Starting with the initial condition $P(T \geq 1) = 1$, (1) can be used to solve for $P(T \geq k)$ for all k . The result is $P(T \geq k) = (1 - \lambda_T(1))(1 - \lambda_T(2)) \cdots (1 - \lambda_T(k-1))$ for $k \geq 1$. Therefore,

$$\begin{aligned} P(T = k) &= P(\{T \geq k\} \cap \{T = k\}) \\ &= P(T \geq k)P(T = k|T \geq k) = P(T \geq k)\lambda_T(k) \\ &= (1 - \lambda_T(1)) \cdots (1 - \lambda_T(k-1))\lambda_T(k). \end{aligned}$$

In particular, if $\lambda_T(k) = p$ for all $k \geq 1$, then $p_T(k) = (1 - p)^{k-1}p$ for $k \geq 1$. In other words, T has the geometric distribution with parameter p .

(e) Fix $k \geq 0$. Consider conditional probabilities given the event $\{T \geq k\}$. Equivalently, consider conditional probabilities given that both links are still working at time k . Then the conditional probability that the network fails at time k , is the conditional probability that the first link fails at time k , plus the conditional probability that the second link fails at time k , minus the conditional probability that both links fail at time k . That is $\lambda_T(k) = \lambda_1(k) + \lambda_2(k) - \lambda_1(k)\lambda_2(k)$. (Note: If $\lambda_1(k)$ and $\lambda_2(k)$ are both very small, then $\lambda_T(k) \approx \lambda_1(k) + \lambda_2(k)$.)

3. Reliability of a self-healing ring communication network

(a) We need to find $P(X \geq 2)$, where X is the number of failed links. The assumptions imply that X has the binomial distribution with parameters $n = 5$ and $p = 0.001$. Thus, using $q = 1 - p$, $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - q^5 - 5pq^4 = 0.00000998 \approx 0.000010$. Another way to do this is $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = 10p^2q^3 + 10p^3q^2 + 5p^4q + p^5 = 0.00000997 + 0.00000001 + 0.000000000005 + 0.00000000000001$

(b) That is $P(X = 2) = 10p^2q^3 = 0.00000997$, (which is very nearly equal to the answer to part (a))

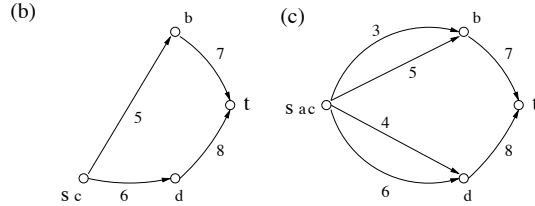
(c) Link 1 must fail, and at least one of links 2, 3, 4, or 5 must fail, which has probability $p[1 - q^4] \approx 0.000004$.

(d) At least one of links 1 or 2 must fail, and at least one of links 3, 4, or 5 must fail, which has probability $[1 - q^2][1 - q^3] \approx 0.000006$. (Note, on a logarithm scale, the answers to parts (a)-(d) are all about the same.)

4. Network reliability problem

(a) $P(F|F_1F_2) = 1$, because every directed $s - t$ path uses either link 1 or link 2.

(b) Given the event $F_1F_2^c$, the original network is equivalent to the one pictured. Node a and links 1, 3, 4 are deleted, because they can't be used, and nodes s and c are merged, because c can always be reached from s . Analysis of the equivalent network yields $P(F|F_1F_2^c) = (2p - p^2)^2 = 0.0361$. Equivalently, with $q = 1 - p$, $P(F|F_1F_2^c) = (1 - q^2)^2$. Similarly, $P(F|F_1^cF_2) = (2p - p^2)^2 = 0.0361$.



(c) Given the event $F_1^cF_2^c$, the original network is equivalent to the one pictured. Nodes s, a , and c are merged, because a and c can always be reached from s . Analysis of the equivalent network yields $P(F|F_1^cF_2^c) = (p^2 + p - p^3)^2 = 0.011881$.

(d) $P(F) = p^2P(F|F_1F_2) + pqP(F|F_1F_2^c) + pqP(F|F_1^cF_2) + q^2P(F|F_1^cF_2^c) = 0.0261$.

(e) The network is tolerant to single link failures, and there are only two possible double link failures that can make the network fail, namely F_1F_2 and F_7F_8 . It is much more likely that either of these two events occurs than it is that three or more links will fail in this small network, for $p = 0.001$. Thus, for $p = 0.001$, $P(F) \approx P(F_1F_2 \cup F_7F_8) \approx P(F_1F_2) + P(F_7F_8) = 0.000002$.

5. A series parallel network

(a) $P(F) = P(F_1) + P(F_2F_3) - P(F_1F_2F_3) = 0.1 + 0.01 - 0.001 \approx 0.109$. Therefore,

$$P(F_1|F) = \frac{P(F_1F)}{P(F)} = \frac{P(F_1)}{P(F)} = \frac{0.1}{0.109} = 0.917$$

(b)

$$P(F_2|F) = \frac{P(F_2F)}{P(F)} = \frac{P(F_2F_3 \cup F_2F_1)}{P(F)} = \frac{0.01 + 0.01 - 0.001}{0.109} = \frac{0.019}{0.109} = 0.1743.$$

6. Packet length choice for noisy channels

- (a) $p_{S,packet} = (1 - p_{e,Byte})^{n+h}$, because a packet is successful if and only if all $n + h$ bytes are correctly received.
- (b) $E[X] = nP[X = n] = np_{S,packet}$, so $\eta = \frac{n}{n+h}p_{S,packet}$. (The term $\frac{n}{n+h}$ accounts for the loss in efficiency due to the packet headers and the term $p_{S,packet}$ accounts for the fraction of packet transmissions that are successful.)
- (c) Very small n is not efficient due to header overhead, reflected in the factor $\frac{n}{n+h}$.
- (d) Very large n is not efficient due to very small packet success probability, reflected by the exponent n in the expression for $p_{S,packet}$.
- (e) Using a small computer program to compute η for a range of n , we find that $n = 53$ maximizes η , though the value of η is nearly the same for all n near 53. The value $n = 53$ yields $\eta \approx 0.151$.

7. A cumulative distribution function

- (a) $P(X \leq 1) = F_X(1) = 0.05$.
- (b) $P(X \leq 10) = F_X(10) = 0.75$.
- (c) $P(X > 10) = 1 - P(X \leq 10) = 0.25$.
- (d) $P(X \geq 10) = P(X > 10) + P(X = 10) = 0.25 + 0.25 = 0.5$.
- (e) Note that $\{|X - 5| \leq 0.1\} = \{4.9 < X \leq 5.1\} \cup \{X = 4.9\}$. There is no jump in the CDF at 4.9, so $P(X = 4.9) = 0$. Thus, $P(|X - 5| \leq 0.1) = P(4.9 < X \leq 5.1) = F_X(5.1) - F_X(4.9) = 0.5 - 0.245 = 0.255$.
- (f) The region mentioned can be partitioned into three rectangles and a triangle, with total area $0.25 \times (15 + 10 + 5) + \frac{1}{2} \times 0.25 \times 5 = 8.125$. This is $E[X]$.