

1. Parity of a binomial random variable, revisited

(a) Since $Y_{n+1} = Y_n + X_{n+1}$, the event G_{n+1} is true if Y_n and X_{n+1} are both even, or both odd. Thus, $G_{n+1} = G_n\{X_{n+1} = 1\}^c \cup G_n^c\{X_{n+1} = 1\}$.

(b) The events comprising the union in part (a) are disjoint, and the outcome X_{n+1} is drawn independently of the first n outcomes. Thus,

$$a_{n+1} = P(G_n\{X_{n+1} = 1\}^c) + P(G_n^c\{X_{n+1} = 1\}) = a_n(1 - p) + (1 - a_n)p$$

or $a_{n+1} = p + (1 - 2p)a_n$.

(c) The recursion in part (b) is a standard first order linear update equation, and the solution can be found in many ways. (One could even guess the solution and check it by induction on n .) Here is one way to find the solution. The recursion can be rewritten as $a_{n+1} - \frac{1}{2} = (1 - 2p)(a_n - \frac{1}{2})$. Along with the initial condition $a_0 - \frac{1}{2} = \frac{1}{2}$, we find that $a_n - \frac{1}{2} = \frac{(1-2p)^n}{2}$, or that $a_n = \frac{1+(1-2p)^n}{2}$.

2. Ternary channel

(a) $P(s_j \text{ received}) = \sum_{i=1}^3 P(s_j \text{ received} | s_i \text{ sent}) \cdot P(s_i \text{ sent})$. Since $P(s_i \text{ sent}) = 1/3$ for all i , the required probabilities above are found to be

	s_1	s_2	s_3
$P(s_j \text{ received})$	0.29	0.36	0.35

Now, we compute $P(s_i \text{ sent} | s_j \text{ received})$. For example, $P(s_1 \text{ sent} | s_1 \text{ received}) = \frac{P(s_1 \text{ received} | s_1 \text{ sent}) \cdot P(s_1 \text{ sent})}{P(s_1 \text{ received})} = \frac{0.8 \cdot 1/3}{0.29} = 0.9195$. In this fashion, we can form a table similar to the one given in the problem, but consisting of the inverse conditional probabilities $P(s_i \text{ sent} | s_j \text{ received})$:

		Receive, j		
		s_1	s_2	s_3
Send, i	s_1	0.920	0.093	0.095
	s_2	0.057	0.8333	0.048
	s_3	0.023	0.074	0.857

3. Randomly selected die

Let E_i be the event that the die with i red faces is drawn from the bag, for $i = 1, 2$, or 3 . Let R_j denote the event that red comes up on the j th role of the die.

(a) $P(R_1) = P(R_1|E_1)P(E_1) + P(R_1|E_2)P(E_2) + P(R_1|E_3)P(E_3) = \frac{1}{6}\frac{1}{3} + \frac{2}{6}\frac{1}{3} + \frac{3}{6}\frac{1}{3} = \frac{1}{3}$

(b) We need to find $P(R_2|R_1)$.

First, $P(R_1R_2) = P(R_1R_2|E_1)P(E_1) + P(R_1R_2|E_2)P(E_2) + P(R_1R_2|E_3)P(E_3) = (\frac{1}{6})^2\frac{1}{3} + (\frac{2}{6})^2\frac{1}{3} + (\frac{3}{6})^2\frac{1}{3} = \frac{7}{54}$.

Therefore, $P(R_2|R_1) = \frac{P(R_1R_2)}{P(R_1)} = \frac{7}{18}$. (Note: we have essentially used Bayes' formula.)

(c) We need to find $P(E_3|R_1R_2R_3)$.

Now $P(E_3R_1R_2R_3) = P(E_3)P(R_1R_2R_3|E_3) = \frac{1}{3}(\frac{3}{6})^3 = \frac{1}{24}$, and

$P(R_1R_2R_3) = P(R_1R_2R_3|E_1)P(E_1) + P(R_1R_2R_3|E_2)P(E_2) + P(R_1R_2R_3|E_3)P(E_3) = (\frac{1}{6})^3\frac{1}{3} +$

$$\left(\frac{2}{6}\right)^3 \frac{1}{3} + \left(\frac{3}{6}\right)^3 \frac{1}{3} = \frac{1}{18}.$$

Therefore, $P(E_3|R_1R_2R_3) = \frac{P(E_3R_1R_2R_3)}{P(R_1R_2R_3)} = \frac{18}{24} = \frac{3}{4}$. Note: we have essentially used Bayes formula.

4. Paging a mobile station, with possible misses

(a) No matter which cell the MS is located in, the first time the MS is paged in the cell in which it is located, it will be missed with probability 0.1. Given that the MS is missed in the first round, it will be missed again in the second round with conditional probability 0.1. So $P(G_8) = (0.1)(0.1) = 0.01$.

(b) Now $P(F_2|G_1) = \frac{P(F_2G_1)}{P(G_1)}$. By the law of total probability with the partition consisting of events E_1, E_2, E_3 , and E_4 ,

$$\begin{aligned} P(G_1) &= P(G_1|E_1)0.4 + P(G_1|E_2)0.3 + P(G_1|E_3)0.2 + P(G_1|E_4)0.1 \\ &= (0.1)(0.4) + (1)(0.3) + (1)(0.2) + (1)(0.1) = 0.64. \end{aligned}$$

and

$$\begin{aligned} P(F_2G_1) &= P(F_2G_1|E_1)0.4 + P(F_2G_1|E_2)0.3 + P(F_2G_1|E_3)0.2 + P(F_2G_1|E_4)0.1 \\ &= (0)(0.4) + (0.9)(0.3) + (0)(0.2) + (0)(0.1) = 0.27. \end{aligned}$$

Thus, $P(F_2|G_1) = \frac{0.27}{0.64} \approx 0.429$

(c) We are seeking $P(F_4|G_3) = \frac{P(F_4G_3)}{P(G_3)}$, where

$$\begin{aligned} P(G_3) &= P(G_3|E_1)0.4 + P(G_3|E_2)0.3 + P(G_3|E_3)0.2 + P(G_3|E_4)0.1 \\ &= (0.1)(0.4) + (0.1)(0.3) + (0.1)(0.2) + (1)(0.1) = 0.19. \end{aligned}$$

and

$$\begin{aligned} P(F_4G_3) &= P(F_4G_3|E_1)0.4 + P(F_4G_3|E_2)0.3 + P(F_4G_3|E_3)0.2 + P(F_4G_3|E_4)0.1 \\ &= (0)(0.4) + (0)(0.3) + (0)(0.2) + (0.9)(0.1) = 0.09. \end{aligned}$$

Thus, $P(F_4|G_3) = \frac{0.09}{0.19} \approx 0.474$.

(d) We are seeking $P(F_8|G_7) = \frac{P(F_8G_7)}{P(G_7)}$, where

$$\begin{aligned} P(G_7) &= P(G_7|E_1)0.4 + P(G_7|E_2)0.3 + P(G_7|E_3)0.2 + P(G_7|E_4)0.1 \\ &= (0.01)(0.4) + (0.01)(0.3) + (0.01)(0.2) + (0.1)(0.1) = 0.019. \end{aligned}$$

and

$$\begin{aligned} P(F_8G_7) &= P(F_8G_7|E_1)0.4 + P(F_8G_7|E_2)0.3 + P(F_8G_7|E_3)0.2 + P(F_8G_7|E_4)0.1 \\ &= (0)(0.4) + (0)(0.3) + (0)(0.2) + (0.09)(0.1) = 0.009. \end{aligned}$$

Thus, $P(F_8|G_7) = \frac{0.009}{0.019} \approx 0.474$. Note: an explanation for why the answer to (d) is the same as the answer to (c) is that if the MS is not found after the first round of four pages, the conditional distribution of which cell the MS is located in is the same as the original distribution.

5. Let's make a deal

(a.i) By Bayes formula,

$$\begin{aligned} P(E_1|F) &= \frac{P(E_1F)}{P(F)} = \frac{P(E_1)P(F|E_1)}{P(E_1)P(F|E_1) + P(E_2)P(F|E_2) + P(E_3)P(F|E_3)} \\ &= \frac{\frac{1}{3} \cdot \frac{1}{2}}{\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 1} = (1/6)/(1/2) = \frac{1}{3} \end{aligned}$$

(Note, we could have deduced that $P(F) = \frac{1}{2}$ by symmetry.)

(a.ii) Similarly, $P(E_3|F) = \frac{P(E_3F)}{P(F)} = \frac{P(E_3)P(F|E_3)}{P(F)} = (1/3)/(1/2) = \frac{2}{3}$. Thus, Dilbert's better choice is to switch.

(b.ii) By Bayes formula,

$$\begin{aligned} P(E_1|G) &= \frac{P(E_1G)}{P(G)} = \frac{P(E_1)P(G|E_1)}{P(E_1)P(G|E_1) + P(E_2)P(G|E_2) + P(E_3)P(G|E_3)} \\ &= \frac{\frac{1}{3} \cdot 1}{\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{1}{2}} = (1/3)/(1/2) = \frac{2}{3} \end{aligned}$$

(Note, we could have deduced that $P(G) = \frac{1}{2}$ by symmetry.)

(a.ii) Similarly, $P(E_3|G) = \frac{P(E_3G)}{P(G)} = \frac{P(E_3)P(G|E_3)}{P(G)} = (1/6)/(1/2) = \frac{1}{3}$. Thus, Dilbert's better choice is to stick, for this version of the game.

6. Let's make a deal-revisited

(a.i) With the "stick" strategy, Dilbert wins on E_1 and loses otherwise, so the probability of winning is $\frac{1}{3}$.

(a.ii) With the "switch" strategy, Dilbert loses on the event E_1 , and wins otherwise, so the probability of winning is $\frac{2}{3}$. The "switch" strategy is better.

(b.i) With the "stick" strategy, Dilbert wins on E_1 , and loses otherwise. So his probability of winning is $\frac{1}{3}$. (This is exactly the same as part (a.i).)

(b.ii) With the "switch" strategy, Dilbert loses on E_1 and E_2 , and wins with probability $\frac{1}{2}$ given the event E_3 . So his overall probability of winning is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$. For this version of the game, the "stick" strategy is better.