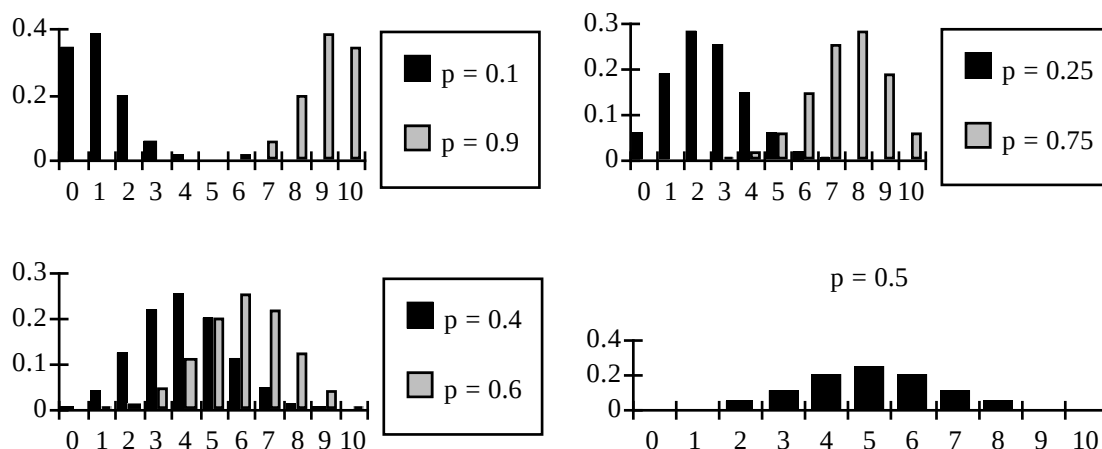


1. (a) (b) The graphs are as shown below.



- (c) The pmfs of X_p and X_{1-p} are “reverses” of each other. For each choice of k , $k = 0, 1, \dots, 10$, and any value of p , $P\{X_p = k\} = P\{X_{1-p} = 10 - k\}$. Note that X_p counts the number of occurrences on 10 independent trials of an event A of probability p . But, if $X_p = k$, then the *complementary event* A^c (of probability $1 - p$), must have occurred $10 - k$ times, and the number of occurrences of A^c is counted by X_{1-p} .
2. (a) X denotes a binomial random variable with parameters (N, p) . It counts the number of occurrences of an event A of probability p on N independent trials. $Y = N - X$ counts the number of occurrences of A^c , an event of probability $1 - p$, on the N independent trials and is thus a binomial random variable with parameters $(N, 1 - p)$.
- (b) $P\{X \text{ is even}\} = \binom{N}{0}(1-p)^N + \binom{N}{2}(1-p)^{N-2}p^2 + \binom{N}{4}(1-p)^{N-4}p^4 + \dots$
Now, a minor extension to Problem 1(e) of Problem Set 2 shows that $(x + y)^N + (x - y)^N = 2 \left[\binom{N}{0}x^N + \binom{N}{2}x^{N-2}y^2 + \dots \right]$, and so, setting $x = 1 - p$, $y = p$, we get $P\{X \text{ is even}\} = \frac{1}{2} [(1 - p + p)^N + (1 - p - p)^N] = \frac{1}{2} [1 + (1 - 2p)^N]$.
Notice that the probability is $1/2$ for $p = 1/2$.
3. X takes on values $0, 1, 2, 3, 4, 5, 6, 7, 8$ with probabilities $p_X(i) = \frac{1}{256}, \frac{8}{256}, \frac{28}{256}, \frac{56}{256}, \frac{70}{256}, \frac{56}{256}, \frac{28}{256}, \frac{8}{256}, \frac{1}{256}$.

(a) $E[X] = \sum_{i=0}^8 i \cdot p_X(i) = 4$ by tedious calculation, or, more simply, $E[X] = np = 4$.

(b) Y takes on values $1, 2, 3, 0$ with probabilities $\frac{28}{256}, \frac{8}{256}, \frac{1}{256}$, and $1 - \frac{28+8+1}{256} = \frac{219}{256}$.

(c) i. $E[Y] = \sum_{i=0}^3 i \cdot p_Y(i) = \frac{47}{256}$. ii. $E[Y] = \sum_{i=0}^8 \max\{X - 5, 0\} \cdot p_X(i) = \frac{47}{256}$

- (d) The airline has collected \$400 in fares from the passengers.
The operating cost is $\$200 + 10(X - Y)$ and the airline pays out $\$70Y$ in returned fares plus penalties. Thus, $\$Z = 400 - 200 - 10(X - Y) - 70Y = 200 - 10X - 60Y$ is the net profit from the flight. Hence, $E[Z] = 200 - 10E[X] - 60E[Y] \approx \$148.98 \dots$
Notice how LOTUS and the linearity of expectation allow us to easily compute $E[Z]$ without finding the pmf of Z first.

4. Each die has probability $\frac{1}{6}$ of matching your chosen number. Thus, the number of dice that match is a binomial random variable with parameters $(3, \frac{1}{6})$.

(a) (b) You lose \$6 if none of the dice show your chosen number. This occurs with probability $(\frac{5}{6})^3 = \frac{125}{216}$. If one of the dice matches, you win \$6 with probability $3 \cdot \frac{1}{6} \cdot (\frac{5}{6})^2 = \frac{75}{216}$. You win \$12 with probability $3 \cdot \frac{5}{6} \cdot (\frac{1}{6})^2 = \frac{15}{216}$, and you win \$18 with probability $(\frac{1}{6})^3 = \frac{1}{216}$. Rationality test: $125 + 75 + 15 + 1 = 216$.

(b) $E[\mathcal{X}] = (-6)\frac{125}{216} + 6\frac{75}{216} + 12\frac{15}{216} + 18\frac{1}{216} = -\frac{102}{216}$.

(c) If all three dice show different numbers (which has probability $\frac{6 \cdot 5 \cdot 4}{6 \cdot 6 \cdot 6} = \frac{120}{216}$ of occurring), you win \$1 on the three numbers but lose \$1 on the other three for a net gain of \$0. If two dice show the same number (which has probability $3 \cdot \frac{6 \cdot 1 \cdot 5}{6 \cdot 6 \cdot 6} = \frac{90}{216}$ of occurring) you win \$2 for that number and \$1 for the other number showing, but lose \$1 on the remaining four for a net loss of \$1. If all three dice show the same number (which has probability $6 \cdot \frac{1 \cdot 1 \cdot 1}{6 \cdot 6 \cdot 6} = \frac{6}{216}$ of occurring), you win \$3 for that number, but lose \$1 on the other five for a net loss of \$2. Thus,

$$E[\mathcal{Y}] = (0)\frac{120}{216} + (-1)\frac{90}{216} + (-2)\frac{6}{216} = -\frac{102}{216}.$$

Thus, the average loss is the same regardless of whether you split your bet or not. Worse yet, your wealth will never increase if you split your bet: you will *never* win any money, whereas if you bet your money on one number, there is a small chance that you will be ahead at some time. (Do remember to quit while you are ahead, will ya?)

5. (a) $P\{\mathcal{Y} \text{ is even}\} = P\{\mathcal{Y} = 0\} + P\{\mathcal{Y} = 2\} + P\{\mathcal{Y} = 4\} + \dots = \exp(-\lambda) \left[1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots \right]$
 $= \exp(-\lambda) \cosh(\lambda)$ by recognizing the series in square brackets.

(b) $\frac{1}{2} [1 + (1 - 2p)^N] = \frac{1}{2} \left[1 + \left(1 - \frac{2\lambda}{N} \right)^N \right] \rightarrow \frac{1}{2} [1 + \exp(-2\lambda)] = \exp(-\lambda) \frac{\exp(\lambda) + \exp(-\lambda)}{2}$
 $= \exp(-\lambda) \cosh(\lambda).$

6. (a) On average, $E[\mathcal{X}] = 105 \times 0.9 = 94.5$ passengers show up for the flight.

(b) $P\{\mathcal{X} \leq 100\} = 1 - P\{\mathcal{X} > 100\}$
 $= 1 - P\{\mathcal{X} = 101\} - P\{\mathcal{X} = 102\} - P\{\mathcal{X} = 103\} - P\{\mathcal{X} = 104\} - P\{\mathcal{X} = 105\}$
 $= 1 - \binom{105}{101}(0.9)^{101}(0.1)^4 - \binom{105}{102}(0.9)^{102}(0.1)^3 - \binom{105}{103}(0.9)^{103}(0.1)^2 - \binom{105}{104}(0.9)^{104}(0.1)^1 - \binom{105}{105}(0.9)^{105}(0.1)^0$
 $= 1 - \binom{105}{4}(0.9)^{101}(0.1)^4 - \binom{105}{3}(0.9)^{102}(0.1)^3 - \binom{105}{2}(0.9)^{103}(0.1)^2 - \binom{105}{1}(0.9)^{104}(0.1)^1 - \binom{105}{0}(0.9)^{105}(0.1)^0 = 0.9832 \dots$

(c) We saw earlier that if $\mathcal{X}$ is a binomial random variable with parameters (n, p) , then $\mathcal{Y} = n - \mathcal{X}$ is a binomial random variable with parameters $(n, 1 - p)$.

(d) $P\{\mathcal{Y} \geq 5\} = 1 - P\{\mathcal{Y} = 0\} - P\{\mathcal{Y} = 1\} - P\{\mathcal{Y} = 2\} - P\{\mathcal{Y} = 3\} - P\{\mathcal{Y} = 4\}$
 $= 1 - \exp(-10.5) \left[1 + \frac{10.5}{1!} + \frac{(10.5)^2}{2!} + \frac{(10.5)^3}{3!} + \frac{(10.5)^4}{4!} \right] = 0.9789 \dots$

7. (a) Your net worth decreases by \$1 with probability $\frac{31}{32}$ and increases by \$28 with probability $\frac{1}{32}$ for an average *loss* of $\$ \frac{3}{32}$ per drawing. A fair bet would have a prize of \$32, not \$29.

(b) Let $\mathcal{X}$ denote the number of the trial on which 7 occurs for the first time. Then, $\mathcal{X}$ is a geometric random variable with parameter $\frac{1}{32}$. Your winnings are $\$ \mathcal{Y} = \begin{cases} \$29 - \mathcal{X}, & \mathcal{X} \leq 29, \\ -\$29, & \mathcal{X} > 29, \end{cases}$ and LOTUS gives $\$E[\mathcal{Y}] = -\$1.80529 \dots$. On average, you lose less than $(29 \times \$ \frac{3}{32})$, because you don't always bet on all 29 draws; you prudently stop when you have won.

(c) $P\{\mathcal{X} > 21\} = \left(1 - \frac{1}{32} \right)^{21} = 0.5133884$, and so you lose \$1 with probability 0.5133884 and win \$1 with probability $1 - 0.5133884 = 0.4866116$ for an average loss of \$0.0267767 per Minibucks ticket.