

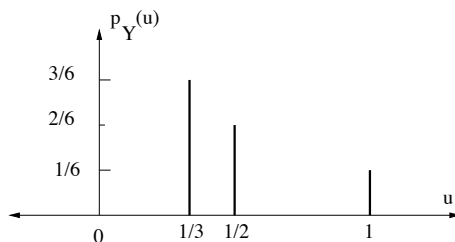
1. Illustration of the law of the unconscious statistician (LOTUS) with a ramp pmf

(a) The function is nonnegative. We thus just have to check that the total mass is one, or equivalently, that $1+2+\cdots+n = n(n+1)/2$. One way to verify this identity is to note that the sum is half as much as $1+2+\cdots+n+n+\cdots+2+1 = (1+n)+(2+n-1)+\cdots+(n+1) = n(n+1)$. Or it can be shown by induction on n .

(b) The possible values of Y are $1, 1/2, 1/3, \dots, 1/n$. We can write

$$p_Y(u) = \begin{cases} \frac{2}{un(n+1)} \text{ (or, equivalently, } \frac{2k}{n(n+1)}) & \text{if } u = 1/k \text{ for some } k \text{ with } 1 \leq k \leq n \\ 0 & \text{else} \end{cases}$$

The pmf of Y for the case $n = 3$ is:



(c) By definition, $E[Y] = \sum_u u p_Y(u)$. There are n nonzero terms in this sum, which each have the form $u \cdot \frac{2}{un(n+1)} = \frac{2}{n(n+1)}$. Thus, all n terms are $\frac{2}{n(n+1)}$, so we have $E[Y] = n \cdot \frac{2}{n(n+1)} = \frac{2}{n+1}$.

(d) Using LOTUS we get, more directly, $E[Y] = \sum_k g(k) p_X(k) = \sum_{k=1}^n \frac{1}{k} \frac{2k}{n(n+1)} = \frac{2n}{n(n+1)} = \frac{2}{n+1}$.

2. Mean, variance, and exponential moments of the Poisson distribution

(a) $\sum_{i=0}^{\infty} \frac{e^{-\lambda} \lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$.

(b) Since, by convention, $0! = 1$ and $\lambda^0 = 1$, we find $P(X = 0) = \frac{e^{-\lambda} \lambda^0}{0!} = e^{-\lambda}$.

(c) $E[X] = \sum_{i=0}^{\infty} i \frac{e^{-\lambda} \lambda^i}{i!}$. The i th term is zero for $i = 0$, so the index i can be taken to run from $i = 1$ to ∞ . For $i \geq 1$, $\frac{i}{i!} = \frac{1}{(i-1)!}$. Therefore, the sum becomes $E[X] = \sum_{i=1}^{\infty} \frac{e^{-\lambda} \lambda^i}{(i-1)!}$. Letting $j = i - 1$, and using the fact $\lambda^i = \lambda \lambda^{i-1} = \lambda \lambda^j$, the sum can be written as $E[X] = \lambda \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^j}{j!} = \lambda$, where for the last equality we used the fact, verified in part (a), that the total mass is one.

(d) Using the LOTUS, we find $E[X(X-1)] = \sum_{i=0}^{\infty} i(i-1) \frac{e^{-\lambda} \lambda^i}{i!}$. The i th term is zero for $i = 0$ and $i = 1$, so those terms can be left out of the summation. For $i \geq 2$, $\frac{i(i-1)}{i!} = \frac{1}{(i-2)!}$.

Therefore, the sum becomes $E[X(X-1)] = \sum_{i=2}^{\infty} \frac{e^{-\lambda} \lambda^i}{(i-2)!}$. Letting $k = i - 2$, and using the fact $\lambda^i = \lambda^2 \lambda^k$, the sum can be written as $E[X(X-1)] = \lambda^2 \sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} = \lambda^2$, where for the last equality we used the fact, verified in part (a), that the total mass is one.

(e) The facts $E[X] = \lambda$ and $E[X^2] - E[X] = \lambda^2$, yield that $E[X^2] = \lambda^2 + \lambda$. Therefore, $\text{Var}(X) = E[X^2] - E[X]^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$. That is, both the mean and variance of X are λ .

(f) $E[z^X] = \sum_{i=0}^{\infty} z^i \frac{e^{-\lambda} \lambda^i}{i!} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{(\lambda z)^i}{i!} = e^{-\lambda} e^{\lambda z} = e^{-\lambda(1-z)}$.

3. Estimation of the parameter of a Poisson random variable

(a) Given n , we wish to select λ to maximize $\frac{\lambda^n e^{-\lambda}}{n!}$. Equivalently, we select λ to maximize $f(\lambda) = \lambda^n e^{-\lambda}$ over $\lambda \geq 0$. If $n = 0$, this function is $e^{-\lambda}$, which is decreasing in λ , so that $\hat{\lambda} = 0$ if $n = 0$. So now suppose $n \geq 1$. In that case, $f'(\lambda) = (n - \lambda)\lambda^{n-1}e^{-\lambda}$. Thus, $f'(\lambda) > 0$ for $\lambda < n$, and $f'(\lambda) < 0$ for $\lambda > n$. Therefore, f is strictly increasing over the interval $[0, n]$ and strictly decreasing over the interval $[n, +\infty)$. Thus, the point $\lambda = n$ (where $f'(\lambda) = 0$) must be the point at which f is maximized. Thus, $\hat{\lambda} = n$ for $n \geq 1$. Combining the cases $n = 0$ and $n \geq 1$, we see that $\hat{\lambda} = n$ for all values of $n \geq 0$.

(b) Given m , we wish to select b to maximize $\frac{(b^2)^m e^{-b^2}}{m!}$. This is the same as maximizing $f(b^2)$ with respect to b , where f is the same as in part (a) with n replaced by m . We know from part (a), that if b is selected so that $b^2 = m$, then $f(b^2)$ is the largest that it can be. Thus $\hat{b} = \sqrt{m}$ is the maximum likelihood estimate of b . (Note, if negative values of b were allowed, $-\sqrt{m}$ would also be a maximum likelihood estimate.)

4. Parameter estimation for an increasing ramp pmf

The problem is to maximize $p_X(k)$ with respect to n , for k fixed. As long as $n \geq k$, this probability is strictly positive, and it is zero otherwise. So the maximizing value, $\hat{n}$, must be greater than or equal to k . Now, for $n \geq k$, the probability is $\frac{2k}{n(n+1)}$, which is strictly decreasing in n . Thus, the maximizing value of n is $\hat{n} = k$.

5. Estimating n from observation of a binomial random variable

(a) Consider a trial to be a placement of one defect, and call a trial a success if the defect is placed in the region examined. Then X is the number of successes in n independent trials, where the probability of success of each trial is $p = 0.25$. So X is a binomial random variable with parameters n and p . More explicitly, the pmf of X is given by $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, for $0 \leq k \leq n$.

(b) Given that $X = k$ (for example, $k = 8$ in the problem statement), we seek the value $\hat{n}$ of n that maximizes $a_n = \binom{n}{k} p^k (1-p)^{n-k}$. Examine the ratio

$$r_n = \frac{a_n}{a_{n-1}} = \frac{n(1-p)}{n-k}$$

for $n \geq k+1$. The ratio r_n is strictly decreasing in n . We can select $\hat{n}$ to be the largest n such that $r_n \geq 1$, or equivalently, the largest n such that $n \leq \frac{k}{p}$, yielding $\hat{n} = \lfloor \frac{k}{p} \rfloor$. If $\frac{k}{p}$ is an integer, then $r_{\frac{k}{p}} = 1$, so in that case there are two values of n that maximize the likelihood: $\frac{k}{p} - 1$ or $\frac{k}{p}$. Thus,

$$\hat{n} = \begin{cases} \lfloor \frac{k}{p} \rfloor & \text{if } \frac{k}{p} \text{ is not an integer} \\ \frac{k}{p} \text{ or } \frac{k}{p} - 1 & \text{if } \frac{k}{p} \text{ is an integer} \end{cases}$$

For the particular problem, either $\hat{n} = 31$ or $\hat{n} = 32$ is a maximum likelihood estimate of n . (Note: If $\frac{k}{p}$ is an integer, the estimator $\lfloor \frac{k}{p} \rfloor$ is equal to $\frac{k}{p}$. Another maximum likelihood estimator is $\lceil \frac{k}{p} - 1 \rceil$, which is equal to $\frac{k}{p} - 1$ if $\frac{k}{p}$ is an integer. The two estimators are equal if $\frac{k}{p}$ is not an integer.)