

1. Since there are 31 *non-empty* subsets of the set of 5 flavors (you wouldn't want to buy or eat unflavored ice-cream, would you?), the manufacturer can create and sell 31 specialty flavors.
2. (a) Since there are $2^7 = 128$ outcomes with $\text{LSB} = 1$, we have that $P(A) = \frac{128}{256} = \frac{1}{2}$.
 (b) There are $\binom{8}{5} = \binom{8}{3} = 56$ outcomes in B and hence $P(B) = \frac{56}{256} = \frac{7}{32}$.
 (c) When the event $A \cap B$ occurs, we know that the 7 more significant bits have exactly 4 ONEs and 3 ZEROes. Hence $|A \cap B| = \binom{7}{4} = \binom{7}{3} = 35$. Thus, $P(A \cap B) = \frac{35}{256}$ and

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{128}{256} + \frac{56}{256} - \frac{35}{256} = \frac{149}{256},$$
 while

$$P(A \oplus B) = P(A) + P(B) - 2 \cdot P(A \cap B) = \frac{128}{256} + \frac{56}{256} - 2 \cdot \frac{35}{256} = \frac{114}{256} = \frac{57}{128}.$$
3. (a) There are $\binom{6}{3} = 20$ possible short lists.
 (b) i. $P(\text{Beth is on the short list}) = \binom{5}{2}/20 = \frac{1}{2}$.
 ii. $P(\text{Chuck is on the short list}) = \binom{5}{2}/20 = \frac{1}{2}$ also.
 iii. $P(\text{Beth and Chuck both are on the short list}) = \binom{4}{1}/20 = \frac{1}{5}$.
 iv. $P(\text{Chuck and two women are on the short list}) = \binom{3}{2}/20 = \frac{3}{20}$.
4. (a) The 0-th derivative $f^{(0)}(x)$ is just $f(x) = (1+x)^n$ itself. The first derivative is $f^{(1)}(x) = n(1+x)^{n-1}$, the second is $f^{(2)}(x) = n(n-1)(1+x)^{n-2}$, and so on. The k -th derivative is $f^{(k)}(x) = n(n-1) \cdots (n-k+1)(1+x)^{n-k}$, and finally the n -th derivative is $f^{(n)}(x) = n(n-1) \cdots 2 \cdot 1 = n!$ which is a constant. Hence, the first $n+1$ terms of the Maclaurin series for $f(x)$ are

$$f(x) = (1+x)^n \approx \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^n \frac{n(n-1) \cdots (n-k+1)}{k!} x^k.$$

- (b) Since $f^{(n)}(x)$ is a constant, $f^{(k)}(x) = 0$ for all $k > n$. Thus, the above is the complete Maclaurin series for $f(x)$; there is *no* approximation.
- (c)
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)(n-k)(n-k-1) \cdots 2 \cdot 1}{((n-k)(n-k-1) \cdots 2 \cdot 1)k!}$$

$$= \frac{n(n-1) \cdots (n-k+1)}{k!}$$
 and so the results match.
- (d) Using the chain rule,

$$\begin{aligned} g^{(k)}(x) &= (-n)(-n-1)(-n-2) \cdots (-n-k+1)(1-x)^{-n-k}(-1)^k \\ &= n(n+1)(n+2) \cdots (n+k-1)(1-x)^{-n-k} \end{aligned}$$

Thus, $g^{(k)}(0) = n(n+1)(n+2) \cdots (n+k-1) \neq 0$ for all integers $k \geq 0$. Hence, the Maclaurin series for $g(x)$ contains terms of all degrees, and we have that

$$g(x) = (1-x)^{-n} = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{n(n+1) \cdots (n+k-1)}{k!} x^k.$$

- (e) For $n = 1$, $n(n+1) \cdots (n+k-1) = k!$ while for $n = 2$, $n(n+1) \cdots (n+k-1) = (k+1)!$. Hence,

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \quad \text{and} \quad (1-x)^{-2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$$

- (f) The k -th derivative of $h(x) = (1+x)^a$ where a is a real number and not necessarily an integer is $h^{(k)}(x) = a(a-1) \cdots (a-k+1)(1+x)^{a-k}$ where the exponent cannot equal 0 except when $a = k$ is a positive integer. Thus, the MacLaurin series has terms of all degrees and we have that

$$h(x) = (1+x)^a = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{a(a-1) \cdots (a-k+1)}{k!} x^k.$$

5. (a) The *complement* of a set of size k is a subset of size $n-k$ and vice versa. Hence, there are as many subsets of size $n-k$ as there are of size k , i.e. $\binom{n}{n-k} = \binom{n}{k}$.

- (b) The first term on the right in $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ is the number of subsets that contain a specific element, say x_1 , with the remaining $k-1$ elements being chosen from the remaining $n-1$ elements. The second term on the right is the number of subsets of size k that *do not* include x_1 .

- (c) $\binom{m+n}{k}$ committees of size k can be drawn from a set of m men and n women. Of these, $\binom{m}{i} \binom{n}{k-i}$ committees have i men and $k-i$ women. Summing over all possible values of i , we conclude that $\binom{m+n}{k} = \sum_{i=0}^k \binom{m}{i} \binom{n}{k-i}$.

- (d) The coefficient of x^k in the polynomial $(1+x)^{m+n}$ is $\binom{m+n}{k}$. Now, if $h(x) = f(x)g(x)$, then $h_k = \sum_i f_i g_{k-i}$. Applying this to $f(x) = (1+x)^m$ and $g(x) = (1+x)^n$, we get that $\binom{m+n}{k} = \sum_{i=0}^k \binom{m}{i} \binom{n}{k-i}$ as before.

- (e) With $m = k = n$,

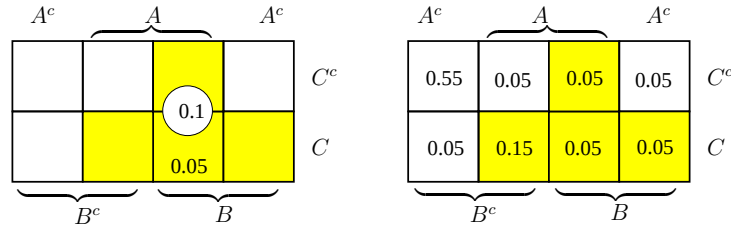
$$\binom{2n}{n} = \sum_{i=0}^n \binom{n}{i} \binom{n}{n-i} = \sum_{i=0}^n \left[\binom{n}{i} \right]^2 \quad \text{since} \quad \binom{n}{n-i} = \binom{n}{i}.$$

- (f)
$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \cdots$$
$$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \binom{n}{4}x^4 - \cdots$$

from which we get that $(1+x)^n + (1-x)^n = 2 \left[\binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \cdots \right]$. Set $x = 1$ and note that the left side has value 2^n while the right side is twice the number of sets with an even number of elements. More simply, set $x = 1$ in $(1-x)^n$ to get

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \Rightarrow \binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}.$$

6. (a) There are $\binom{5}{2} = 10$ games in this tournament, and $\binom{n}{2} = \frac{n(n-1)}{2}$ games in general.
- (b) Yes. In fact, many schedules can be specified. Here is one: imagine that the teams have been arranged in a circle so that each team can be thought of as having two teams on its left and two teams on its right. Then, each team wears home uniforms when playing a team on its left and away uniforms when playing a team on its right.
- (c) There are $2^{10} = 1024$ possible outcomes of this tournament.
- Two teams cannot possibly have 4-0 records. However, if Team A, say, has a 4-0 record, then we know what happened in 4 games, while the outcomes of the remaining 6 games are arbitrary. Hence, $P(\text{Team A has a 4-0 record}) = \frac{2^6}{2^{10}} = \frac{1}{2^4} = \frac{1}{16}$ and $P(\text{some team has a 4-0 record}) = 5 \times \frac{1}{16} = \frac{5}{16}$. (What axiom are we using here?)
 - The same argument shows that $P(\text{some team has a 0-4 record}) = \frac{5}{16}$.
 - If Team A wins all four of its games *and* Team B loses all four of its games, then we know what happened in 7 games (why not 8?), and hence we get that $P(\text{Team A has 4-0 record; Team B has a 0-4 record}) = \frac{2^3}{2^{10}} = \frac{1}{2^7} = \frac{1}{128}$, and $P(\text{some team has 4-0 record; some other team has a 0-4 record}) = 20 \times \frac{1}{128} = \frac{5}{32}$.
 - The remaining teams have lost one game (against the 4-0 team) and won another game (against the 0-4 team), and will have identical 2-2 records if each wins one game and loses one game among the three games that these teams play against one another (e.g. A beats B who beats C who beats A: basketball is not necessarily a transitive game!). Since only 2 of the 8 outcomes of these three games give 2-2 records for all three teams, we get that $P(\text{one team is 4-0; another is 0-4; rest are 2-2}) = \frac{5}{32} \times \frac{2}{8} = \frac{5}{128}$.
7. (a) The Karnaugh map is as shown in the left hand figure below, with some probabilities marked on it. Note that the shaded region is the event $(A \cap B) \cup (B \cap C) \cup (A \cap C)$.



- (b) Since $A \cap B$ is the disjoint union of $A \cap B \cap C$ and $A \cap B \cap C^c$, we get that $P(A \cap B) = 0.1 = P(A \cap B \cap C) + P(A \cap B \cap C^c) = 0.05 + P(A \cap B \cap C^c)$ giving that $P(A \cap B \cap C^c) = 0.05$. Since $P(AB \cup BC \cup AC) = 0.3 = P(AB) + P(AB^cC) + P(A^cBC)$ while $P(AC) = P(ABC) + P(AB^cC) = 2P(BC) = P(ABC) + P(A^cBC)$, we readily obtain that $P(AB^cC) = 0.15$ and $P(A^cBC) = 0.05$. Since $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$ we have that $P(A^cB^c) = 1 - P(A \cup B) = 0.4$. Since $P(A^cB^cC) = P(C) - P(BC) - P(AB^cC) = 0.05$, we get that $P(\text{cereal snaps, crackles, and pops}) = P(A^cB^cC) = 0.05$. See the figure on the right.
- (c) $P(\text{the sample fails only the snap test}) = P(AB^cC^c) = 0.05$.
 $P(\text{the sample fails only the crackle test}) = P(A^cBC^c) = 0.05$.
 $P(\text{the sample fails only the pop test}) = P(A^cB^cC) = 0.05$.