

Poisson random processes, Gaussian random variables

Assigned reading: Ross, Sections 5.1-5.5, and 9.1

Noncredit exercises: Ross, Chapter 5: problems 15-34, theoretical exercises 7-14, self-test 5-14, and Ross, Chapter 9, theoretical exercises 1-4, self test 1&2.

1. Moments of the exponential distribution

Suppose X is exponentially distributed with parameter $\mu > 0$.

- (a) Use LOTUS and integration by parts to express $E[X^n]$ in terms of μ , n and $E[X^{n-1}]$ for $n \geq 1$.
- (b) Use part (a) to express $E[X^n]$ in terms of μ and n .
- (c) Use LOTUS to find $E[e^{\alpha X}]$ for $0 < \alpha < \mu$.
- (d) Here is a way to derive the answer to part (b) using part (c). Obtain a power series of $E[e^{\alpha X}]$ with n^{th} term of the form $c_n \alpha^n E[X^n]$, for $n \geq 0$. (Hint: Begin with a power series for $e^{\alpha X}$ and take expectations on both sides.) Expand your answer to part (c) as a power series in α . The series should be convergent for $0 < \alpha < \mu$. Equate the coefficients of the two power series to rederive your answer to part (b).

2. Splitting property of the Poisson distribution

Suppose a coin shows “heads” with probability p , each time it is flipped, and suppose the coin is flipped N times, where N is a Poisson random variable with parameter $\mu > 0$. Let N_1 denote the number of times “heads” shows up, and N_2 denote the number to times “tails” shows up.

- (a) Find $P(N_1 = n_1 | N = n)$ for $0 \leq n_1 \leq n$.
- (b) Find $P(N_1 = n_1, N_2 = n_2)$ (Hint: It is equal to $P(N_1 = n_1, N = n_1 + n_2)$).
- (c) Show that $P(N_1 = n_1, N_2 = n_2)$ can be factored into a function of n_1 and a function of n_2 , where each factor is a pmf. (This result implies that N_1 and N_2 are independent random variables: i.e. $P(N_1 = n_1, N_2 = n_2) = p_{N_1}(n_1)p_{N_2}(n_2)$, and it also identifies the pmfs of N_1 and N_2 . Of course, N_1 and N_2 are not conditionally independent given $N = n$.)

3. Arrival times for a Poisson process

Calls arrive to a cell in a certain wireless communication system according to a Poisson process with arrival rate $\mu = 2$ calls per minute. Measure time in minutes and consider an interval of time beginning at time $t = 0$. Let $N(t)$ denote the number of calls that arrive up until time t .

- (a) What is the pmf of $N(t)$ for a fixed $t > 0$?
- (b) Find the probabilities of each of the events:
 - E_1 = “No calls arrive in the first 3.5 minutes” E_2 = “The first call arrives after time $t = 3.5$ ”
 - E_3 = “2 or fewer calls arrive in the first 3.5 minutes” E_4 = “the third call arrives after time $t = 3.5$ ”
 - E_5 = “the third call arrives after time t ” (For general $t > 0$) (use only finitely many terms)
 - E_6 = “the third call arrives before time t ” (for general $t > 0$) (Hint: $P(E_6) = 1 - P(E_5)$)
- (c) Note that, as a function of t , $P(E_6)$ is the cumulative probability distribution function for the time of the third call arrival. Differentiate this function with respect to t to find the probability density function for the time of the third arrival. Terms cancel, so simplify your answer. You should get the gamma density with parameters $(3, \mu)$.
- (d) What is the expected arrival time of the tenth call?

4. Poisson process probabilities

Consider a Poisson process with rate $\mu > 0$.

- (a) Find the probability that there is (exactly) one arrival in each of the three intervals $(0,1]$, $(1,2]$, and $(2,3]$.
- (b) Find the probability that there are two arrivals in the interval $(0,2]$ and two arrivals in the interval $(1,3]$. (Note: your answer to part (b) should be larger than your answer to part (a)).
- (c) Find the probability that there are two arrivals in the interval $(1,2]$, given that there are two arrivals in the interval $(0,2]$ and two arrivals in the the interval $(1,3]$.

5. Calculating probabilities of events involving a Gaussian random variable

Let X be a $N(10, 16)$ random variable (i.e. a Gaussian random variable with mean 10 and variance 16). Find the numerical values of the following probabilities.

- (a) $P(X \geq 15)$ (b) $P(X \leq 5)$ (c) $P(X^2 \geq 400)$ (d) $P(X = 2)$.

6. Normal approximation to the binomial distribution

Do problem 20, p. 249 of *Ross* using the binomial approximation to the normal distribution, with the continuity correction.