

Continuous random variables

Due: 3/14/07 at the beginning of class

Reading: Chapter 5 of the textbook

Noncredit Exercises: Chapter 5, Problems 1-3, 5-7, 10, 13; Theoretical Exercises 1, 8; Self-Test Problems and Exercises 1-4.

Problems:

1. The expectation of a nonnegative random variable $\mathcal{X}$ is

$$\mathbb{E}[\mathcal{X}] = \int_0^\infty [1 - F_{\mathcal{X}}(u)] du = \int_0^\infty P\{\mathcal{X} > u\} du.$$

- (a) Use this result to prove that if $\mathcal{X}$ is a discrete random variable that takes on nonnegative *integer* values, then $\mathbb{E}[\mathcal{X}] = \sum_{k=0}^{\infty} P\{\mathcal{X} > k\} = \sum_{i=1}^{\infty} P\{\mathcal{X} \geq i\}$. (cf. Theoretical Exercise 6, p. 197 of Ross).

Hint: Break up the integral into the sum of integrals over the unit-length intervals $[k, k+1)$, $k = 0, 1, 2, \dots$

- (b) For $k = 0, 1, 2, \dots$, find $P\{\mathcal{X} > k\}$ for a *geometric* random variable with parameter p . Use these results together with the result of part (a) to provide a different proof of the fact that $\mathbb{E}[\mathcal{X}] = p^{-1}$.
2. Nine functions $f(u)$ are shown below. Note that in each case, $f(u) = 0$ for all u not in the interval specified. In each case,
- determine whether $f(u)$ is a valid probability density function (pdf).
 - If $f(u)$ is not a valid pdf, determine if there exists a (possibly negative) constant C such that $C \cdot f(u)$ is a valid pdf.
- (a) $f(u) = 2u$, $0 < u < 1$. (b) $f(u) = |u|$, $|u| < \frac{1}{2}$
(c) $f(u) = 1 - |u|$, $|u| < 1$, (d) $f(u) = \ln u$, $0 < u < 1$. Hint: $\ln u$ can be integrated by parts.
(e) $f(u) = \ln u$, $0 < u < 2$, (f) $f(u) = \frac{2}{3}(u - 1)$, $0 < u < 3$,
(g) $f(u) = \exp(-2u)$, $u > 0$. (h) $f(u) = 4\exp(-2u) - \exp(-u)$, $u > 0$,
(i) $f(u) = \exp(-|u|)$, $|u| < 1$,
3. Problem 4 on page 247 of Ross.
4. $\mathcal{Y}$ denotes a random variable with probability density function

$$f_{\mathcal{Y}}(v) = \begin{cases} 1 + v, & -1 \leq v \leq 0, \\ v, & 0 < v \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $P\left\{|\mathcal{Y}| < \frac{1}{2}\right\}$, $P\left\{\mathcal{Y} > 0 \mid \mathcal{Y} < \frac{1}{2}\right\}$, and $\mathbb{E}[\mathcal{Y}]$.

5. The weekly demand (measured in thousands of gallons) for gasoline at a rural gas station is a random variable X with probability density function

$$f_X(u) = \begin{cases} 5(1-u)^4, & 0 \leq u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let C (in thousands of gallons) denote the capacity of the tank (which is re-filled weekly.)

- (a) If $C = 0.5$, what is the probability that the weekly demand for gasoline can be satisfied? Note that if your answer is (say) 0.666..., then, in the long run, the gas station can supply the weekly demand two weeks out of three.
- (b) What is the minimum value of C required to ensure that the probability that the demand exceeds the supply is no larger than 10^{-5} ?
- (c) Suppose that the owner makes a gross profit of \$0.64 for each gallon of gasoline sold. Let $\mathcal{Y}$ denote the amount of gasoline sold per week. How is $\mathcal{Y}$ related to $\mathcal{X}$, the weekly demand for gasoline? (Hint: the owner cannot sell more gasoline each week than the tank can hold!) What is the *average* weekly gross profit?
- (d) Suppose that the owner pays $\$20C$ as weekly rent on a tank of capacity $1000C$ gallons. Note that $0 \leq C \leq 1$. (Why is a tank larger than 1000 gallons not needed?) What is the average weekly net profit and what value of C maximizes the average weekly net profit?