

System Reliability and CDFs

Assigned reading: Section 4.9, and review Chapters 3 and the rest of Chapter 4.

Noncredit exercises: Chapter 4, problems 17 and 19, theoretical exercises 17,21,22,23, and even numbered self-test problems and exercises.

1. The reliability of multiple connections

A certain Internet service provider in a midsize city relies on k separate connections between the city and neighboring cities, to connect its customers to the Internet. Based on past experience, management assumes that a given connection will be down on a given day with probability $p = 0.001$, independently of what happens on other days or with other connections. Total outage is said to occur if all connections are down on the same day. How large should k be so that the probability total outage occurs at least one day in a year is less than or equal to 0.001?

2. Failure rate function for discrete random variables

Suppose an item, such as a light bulb, a sensor, or a satellite, will eventually fail. Suppose the time of failure, in some appropriate units of time, is modeled as a random variable T , with values in the set of positive integers. The *failure rate function* λ_T is the function $\lambda_T = (\lambda_T(k) : k \geq 1)$, defined by $\lambda_T(k) = P(T = k | T \geq k)$ if $P(T \geq k) > 0$. If $P(T \geq k) = 0$, then $\lambda_k(T)$ is not defined.

(a) Give a general expression for λ_T in terms of the pmf, p_T .

(b) Note that if $\lambda_T(k)$ is defined for a given value of k , then its value is in the interval $[0, 1]$ (this is true for any conditional probability). What does it mean about the distribution of T if $\lambda_T(k_f) = 1$ for some value $k_f \geq 1$?

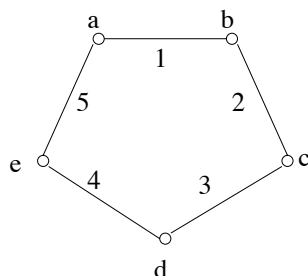
(c) Find λ_T explicitly in case $p_T(k) = \frac{1}{n}$ for $1 \leq k \leq n$. (Hint: $\lambda_T(k)$ is not well defined for $k \geq n + 1$ for this example.)

(d) Find the pmf p_T in case $\lambda_T(k) = p$ for all $k \geq 1$, where p is a constant with $0 < p \leq 1$. (Hint: begin by finding $p_T(1)$ and $p_T(2)$.)

(e) A network consists of two links in series. Let T be the time the network fails, which is the first time that at least one of the links fails. Assume the two links fail independently, with the failure rate function of link i being $\lambda_i = (\lambda_i(k) : k \geq 1)$, for $i = 0$ or $i = 1$. Express the failure rate function of the network, λ_T , in terms of the functions λ_1 and λ_2 . (Hint: Make sure $\lambda_T(k)$ is in the interval $[0, 1]$ if $\lambda_1(k)$ and $\lambda_2(k)$ are both defined. You should also have $\lambda_T(k) = 1$ if $\lambda_i(k) = 1$ for $i = 1$ or $i = 2$.)

3. Reliability of a self-healing ring communication network

Consider the ring network shown, with nodes a, b, c, d , and e , and links numbered 1 through 5.

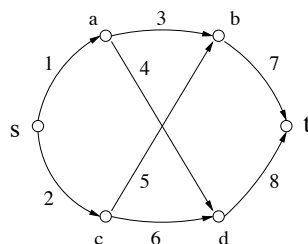


The links are full duplex, meaning they can be used in both directions simultaneously, but each link is assumed to fail in a given time period with probability $p = 0.001$. A pair of nodes is said to be *connected* if they are connected by at least one path with no failed links. The network is said to be *connected* if all pairs of nodes are connected. Find the probabilities of the following events:

- (a) the network is *not* connected
- (b) exactly two links fail
- (c) nodes a and b are *not* connected
- (d) nodes a and c are *not* connected

4. Network reliability problem

Consider the graph shown, with a source node s and terminal node t , intermediate nodes a, b, c , and d , and one-way links numbered 1 through 8.

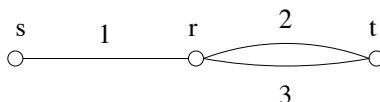


Each link is assumed to fail with probability $p = 0.1$, independently of the other links. The network is said to fail if every directed (i.e. following the directions of the arrows) path from s to t has at least one failed link. Let F_i denote the event that link i fails, and let F denote the event the network fails.

- (a) Find $P(F|F_1F_2)$.
- (b) Find $P(F|F_1F_2^c)$ and $P(F|F_1^cF_2)$.
- (c) Find $P(F|F_1^cF_2^c)$.
- (d) Using the answers to (a)-(c), compute $P(F)$.
- (e) If instead, $p = 0.001$, then $P(F)$ is very close to 0.000002. Why is this expected? (In fact, $P(F) = 0.000002007980$).

5. A series parallel network

Consider a network as shown, with source node s , relay node r , terminal node t , and links 1-3.



Suppose each link fails with probability 0.1, and links fail independently of each other. Suppose the network fails if link 1 fails, or if links 2 and 3 fail, or both. Let F denote the event that the network fails, and F_i denote the event that link i fails. Find (a) $P(F_1|F)$ and (b) $P(F_3|F)$.

6. Packet length choice for noisy channels

Realistically, communication links such as wireless communication links aren't totally predictable. Things such as latency and the fraction of bits that are correctly received vary with time, network load and other variables. A good approach to model such variability is to use probability theory. In the following we explore how the probability of byte error might affect the packet size you use for an application requiring reliable transfer of a large amount of data. Use the following assumptions. Suppose that each packet consists of $h + n$ bytes. The first h bytes of a packet comprise the header, and the remaining n bytes comprise the data bytes. The header contains information such as the destination address, a sequence number, some parity bytes, and so on. Suppose that the probability a byte is received in error is $p_{e,Byte}$, and that the error events for different bytes are independent. Assume that a packet is successfully received if all $n + k$ bytes are received correctly, and that a packet is detected to be in error otherwise. (Packets with errors have to be retransmitted.) (a) Express the probability a packet is successfully received, $p_{S,packet}$, in terms of n , h and $p_{e,Byte}$.

(b) Let X denote the number of data bytes successfully received as the result of a single packet transmission. Possible values of X are 0 or n . The efficiency η is defined to be $E[X]$ divided by $n + h$, which is the number of data bytes successfully received per byte transmitted. Find η as a function of n , h , and $p_{e,Byte}$.

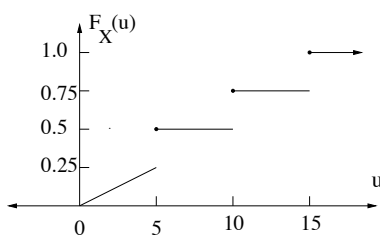
(c) Why is η small if n is very small, and h is fixed?

(d) Why is η small if n is very large, and h is fixed?

(e) Assuming that $h = 60$ bytes and $p_{e,Byte} = 0.01$ (a fairly noisy channel), find the value of the positive integer n to maximize η and find the resulting value of η .

7. A cumulative distribution function

Let X be a random variable with the cumulative distribution (CDF) function shown.



Find the numerical values of the following quantities: (a) $P(X \leq 1)$, (b) $P(X \leq 10)$, (c) $P(X > 10)$, (d) $P(X \geq 10)$, (e) $P(|X - 5| \leq 0.1)$, and (f) $E[X]$, which equals the area of the region to the right of the vertical axis, between the graph of F_X and the horizontal line through one.