

So many choices, so little time ...

This problem set gives you practice in counting and in the calculation of probabilities using the axioms of probability. It also provides more drill in calculus via a derivation of the binomial theorem.

Noncredit Exercises: Do NOT turn these in

Chapter 1: Problems 1-5, 7, 9;

Theoretical Exercises 4, 6, 13; Self-Test Problems 1-15.

Chapter 2: Problems 3, 4, 9, 10, 11-14;

Theoretical Exercises 1-3, 6, 10, 11, 16, 19, 20; Self-Test Problems 1-8

1. An ice cream manufacturer makes unflavored ice cream and then creates “specialty flavors” by blending in one or more of the five essences: vanilla, chocolate, fudge, mint, and almond into the ice cream. How many specialty flavors can the manufacturer create? **Optional noncredit exercise:** identify the manufacturer!
2. An experiment consists of observing the contents of an 8-bit register. We assume that all 256 byte values are equally likely to be observed.
 - (a) Let A denote the event that the least significant bit is a ONE. What is $P(A)$?
 - (b) Let B denote the event that the register contains 5 ONES and 3 ZEROes. What is $P(B)$?
 - (c) What is $P(A \cup B)$? What is $P(A \cap B)$? What is the probability that exactly one of the two events A and B occurs, i.e. what is $P(A \oplus B)$?
3. Some years ago, the government of a small island nation in the North Atlantic Ocean decided to change from a constitutional hereditary monarchy to a constitutional appointed monarchy. Six applicants Andy, Beth, Chuck, Di, Eddie, and Fergie presented themselves before the House of Commons Search Committee for interviews for the job. Subsequently, the Committee forwarded a *short list* of three names to the whole House of Commons for its decision.
 - (a) How many different short lists could the Committee have selected?
 - (b) Assume that all short lists were equally likely to have been chosen.
 - i. What is the probability that Beth is on the short list?
 - ii. What is the probability that Chuck is on the short list?
 - iii. What is the probability that both Beth and Chuck are on the short list?
 - iv. What is the probability that Chuck and two women are on the short list? (Those who wish to pretend that they don’t read the National Enquirer are reminded that Beth, Di, and Fergie are women).
4. (a) For $0 < k \leq n$, compute the k -th derivative of $f(x) = (1+x)^n$ using the chain rule, that is, *without* multiplying out the terms to get a polynomial in x and then differentiating. (In other words, don’t do $\frac{d}{dx}(1+x)^2 = \frac{d}{dx}(1+2x+x^2) = 2+2x$.) Use these derivatives to find the first $n+1$ terms of the MacLaurin series (Taylor series in the vicinity of 0) for $(1+x)^n$.

- (b) Repeat part (a) for $k > n$ and thus find the complete Maclaurin series for $f(x)$.
 (c) According to the textbook (Equation 4.2 in Chapter 1 with $y = 1$),

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Does your answer to part (b) match this result? If so, congratulations! You have just proved the binomial theorem for positive integer exponents.

- (d) Now consider the function $g(x) = (1-x)^{-n}$ where n is a positive integer. Does the Maclaurin series for $g(x)$ contain terms of degree $> n$? If so, what is the term for degree $n+1$? If not, what is the highest degree term?
 (e) Use the result of part (d) to write down the Maclaurin series for $(1-x)^{-1}$ and $(1-x)^{-2}$. These two results will be needed so often in ECE 413 that it is recommended that you memorize them.
 (f) Find the Maclaurin series for $(1+x)^a$ where a is not necessarily an integer.
5. One definition of the binomial coefficient $\binom{n}{k}$ is that it is the number of subsets of size k drawn from a set of size n . Parts (a)-(c) should be solved purely in terms of the definition as the number of subsets, and WITHOUT recourse to the formula for the binomial coefficient shown in Problem 1(c) above (or simplifications thereof by cancelling common terms from numerator and denominator).

- (a) Show that $\binom{n}{k} = \binom{n}{n-k}$. Hint: the *complement* of a set of size k is of size...?

- (b) Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Hint: How many subsets of size k include a specific element? How many don't?

- (c) Show that $\binom{m+n}{k} = \sum_{i=0}^k \binom{m}{i} \binom{n}{k-i}$. Hint: How many committees of size k can be formed from a group of m men and n women?

- (d) What is the coefficient of x^k in the polynomial $(1+x)^{m+n}$?
 Now write $(1+x)^{m+n} = (1+x)^m(1+x)^n$ and find the coefficient of x^k on the right hand side in terms of the coefficients of $(1+x)^m$ and $(1+x)^n$. If you have taken ECE 410, you might recognize a discrete convolution here....

- (e) Use the result of part (c) to show that $\binom{2n}{n} = \sum_{i=0}^n \left[\binom{n}{i} \right]^2$.

- (f) Write down the first five terms in the expansions of $(1+x)^n$ and $(1-x)^n$ (preferably on two consecutive lines and with coefficients of x^k on the two lines being one above the other in vertical alignment) and use these to find formulas for $(1+x)^n + (1-x)^n$ and $(1+x)^n - (1-x)^n$.

Now, a sample space of size n has 2^n subsets (including the empty set $\emptyset$ and the entire space Ω .) Show that exactly 2^{n-1} of the subsets contain an even number of elements (the other 2^{n-1} subsets contain an odd number of elements).

Hint: remember that 0 is an even number and set $x = 1$ in $(1+x)^n \pm (1-x)^n$.

6. Five basketball teams (call them A, B, C, D, and E) play a round-robin tournament in which each team plays each of the other four exactly once.
- (a) How many basketball games are there in this tournament? More generally, how many games would there be in an n -team round-robin tournament?
 - (b) Is it possible to arrange matters so that each of the five teams wears dark-colored (away) uniforms for two games and light-colored (home) uniforms for two games? If so, exhibit a schedule of games specifying the colors worn by each team.
Basketball nonfanatics are reminded that in every basketball game, one team wears dark-colored uniforms and the other wears light-colored uniforms.
 - (c) Now suppose that each game is equally likely to end in a win for either team.
 - i. What is the probability that some team wins all four of its games?
 - ii. What is the probability that some team loses all four of its games?
 - iii. What is the probability that one team wins all four of its games *and* another loses all four of its games?
 - iv. What is the probability that one team wins all four of its games *and* another loses all four of its games *and* the remaining teams have identical 2-2 records?
7. The manufacturer of a cereal tests samples from the production line to see if the samples snap, crackle, and pop as advertised. Let A , B , and C respectively denote the events that the sample being tested *does not* snap, *does not* crackle, and *does not* pop. The manufacturer's tests show that $P(A) = P(C) = 0.3$, $P(B) = 0.2$, $P(AB \cup AC \cup BC) = 0.3$, $P(A \cap B \cap C) = 0.05$, $P(A \cap B) = 0.1$, and $P(A \cap C) = 2P(B \cap C)$.
- (a) Sketch the sample space and indicate on it the events A , B , and C .
 - (b) What is the probability that the cereal snaps, crackles, and pops?
 - (c) Cereal that fails exactly one test is sold to discount supermarket chains to be marketed under the names Soggies, Bleccies, and Mushies. What is the probability that the sample fails *only* the snap test? *only* the crackle test? *only* the pop test?