

## Calculus tune up

**Assigned reading:** Review a calculus book for help with this problem set. Read the following material to help you keep up with the lectures: *Ross* Chapter 1.1–1.4, Chapter 2.1–2.5.

**Noncredit exercises:** Chapter 1, problems 1-5,7,9; Theoretical exercises 4,8,13; Self-test problems 1-15; Chapter 2: problems 3,4,9,10, 11-14; Theoretical exercises 1-3,6,7,10,11,12,16,19,20; Self-test problems 1-8.

**About the credit problems (below):** Calculus, a prerequisite of this course, will be used mainly in the second half of the semester. The parts of calculus primarily needed are integration and differentiation, Taylor series, l'Hôpital's rule, integration by parts, and double integrals (setting up limits of integration and change of variables such as in rectangular to polar coordinates). This problem set will help you review these topics, and identify areas in which you may need additional review.

### 1. Geometric series

- (a) Prove that  $1 + x + x^2 + \dots + x^{n-1} = \frac{1-x^n}{1-x}$  for  $x \neq 1$  and integer  $n \geq 1$ .  
(b) Assuming  $|x| < 1$ , find a simple expression for the sum  $1 + x + \dots$ . (By definition, this sum is the limit of the sum  $1 + x + x^2 + \dots + x^{n-1}$  as  $n \rightarrow \infty$ .)

### 2. Inverse function

Let  $f$  be the function defined by  $f(x) = \frac{2}{e^x + 1}$  for  $-\infty < x < \infty$ . Let  $g$  denote the inverse function of  $f$ , so that  $g(f(x)) = x$  for all  $x$ . Find  $g$ . (Hint: The domain of  $g$  can be smaller than the whole real line. Draw a picture.)

### 3. Extrema of functions

- (a) Find the maxima (plural) of  $f(x) = x^4(1.001)^{-x^2}$  on the real line. (b) Does the function  $f(x) = x^4(1.001)^{x^2}$  have a maximum value at some finite  $x$ ? Explain.

### 4. Some definite integrals

Find the values of the following **definite** integrals:

- (a)  $\int_0^2 \max\{x, 2x - 1\} dx;$       (b)  $\int_0^1 x(1 - x^2)^{11} dx;$       (c)  $\int_0^1 x^2 \exp(-x) dx;$   
(d)  $\int_{-1}^1 x^3 \exp(-x^2/2) dx.$  (Hint: Can solve by inspection. Make a sketch if needed.)

## 5. Derivatives and integrals

Let  $\frac{d}{dx}f(x) = g(x)$ ,  $-\infty < x < \infty$ . Which of the following statements are true for all  $x$ ? (Here,  $C$  denotes an arbitrary constant.)

- (a)  $\frac{d}{dx}f(-x) = -g(-x)$       (b)  $\frac{d}{dx}f(x^2/2) = xg(x^2/2)$       (c)  $\frac{d}{dx}\exp(f(x^2)) = g(x^2)\exp(f(x^2))$   
(d)  $\int g(-x) dx = f(-x) + C$       (e)  $\int g(x^2/2) dx = \frac{f(x^2/2)}{x} + C$       (f)  $\int \frac{g(x)}{f(x)} dx = \ln(f(x)) + C$

## 6. Double integrals

Evaluate the following **definite** two-dimensional integrals over the specified domains of integration:

1.  $f(x, y) = \max(x, y)$ , over the region  $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ .
2.  $f(x, y) = (x^2 + y^2)^{-4}$ , over the region  $\{(x, y) : x^2 + y^2 > 1\}$ .

## 7. An application of Taylor's approximation

(a) Let  $f(x) = \ln(1+x)$ . Find the degree one polynomial  $P(x) = f(0) + f'(0)x$ . (This is the degree one Taylor polynomial of  $f$  centered at zero. Equivalently, it consists of the first two terms of the Maclaurin expansion of  $f$ , because Maclaurin expansions are Taylor expansions centered at zero.)

(b) Find a number  $M$  so that  $|f''(x)| \leq M$  for  $|x| \leq 0.5$ .

(Hint for next part: It follows by Taylor's inequality that  $|f(x) - P(x)| \leq \frac{Mx^2}{2!}$  for  $|x| \leq 0.5$ .)

(c) Given any real value  $a$ , find a simple expression for  $\lim_{n \rightarrow \infty} n \ln(1 + \frac{a}{n})$ .

(d) Find a simple expression for  $\lim_{n \rightarrow \infty} (1 + \frac{a}{n})^n$ . (Hint:  $(1 + \frac{a}{n})^n = \exp(n \ln(1 + \frac{a}{n}))$ .)