Solutions to Exam I **ECE 413, Spring 2007**

Problem 1 *(36 points)* An experiment consists of rolling three fair dice. The rolls are independent trials. Let A be the event that the numbers showing on the three dice are all even, and let B be the event that the numbers showing on the three dice are different from each other. (a) Express the set of possible outcomes Ω using mathematical set notation. (b) Find $P(A)$. (c) Find $P(B)$. (d) Find $P(AB)$. (e) Find $P(A \cup B)$. (f) *Sketch and carefully label* the probability mass function of X, where X denotes the number of distinct numbers showing on the dice.

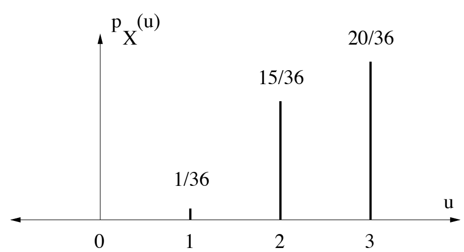
(a) $\Omega = \{x_1x_2x_2 : 1 \le x_i \le 6 \text{ for } 1 \le i \le 3\}$, or $\Omega = \{111, 112, 113, 114, 115, 116, 121, 122, \ldots, 664, 665, 666\}$, or $\Omega = \{1, 2, 3, 4, 5, 6\}^3$.
(b) Since each die shows even with probability $\frac{1}{2}$ and the dice are rolled as independent trials, they all show even with probability $(\frac{1}{2})^3 = \frac{1}{8}$. Another solution is to note that $|A| = 3^3$, $\Omega = 6^3$, and all outcomes in Ω are equally likely, so $P(A) = \frac{|A|}{|\Omega|} = \frac{3^3}{6^3} = \frac{1}{8}$.

(c) $|B| = 6 \cdot 5 \cdot 4$ because to specify an outcome in B, there are six choices for the first die, then five for the second, and then four for the third. So $P(B) = \frac{6\cdot5\cdot4}{6^3} = \frac{20}{36} = \frac{5}{9}$.

(d) $|AB| = 3 \cdot 2 \cdot 1 = 6$, because to specify an outcome in AB, there are three choices for the first die, then two for the second, and then only one for the third. Or we could simply write $AB = \{246, 264, 426, 462, 624, 642\}$. Therefore, $P(AB) = \frac{6}{6^3} = \frac{1}{36}$.

(e) $P(A \cup B) = P(A) + P(B) - P(AB) = \frac{1}{8} + \frac{5}{9} - \frac{1}{36} = \frac{9+40-2}{72} = \frac{47}{72}$.

(f) X takes values in $\{1, 2, 3\}$. Now, $p_X(1) = P(\{111, 222, 333, 444, 555, 666\}) = \frac{6}{6^3} = \frac{1}{36}$. (Another way to see this is that, no matter what shows on the first die, chances are $\frac{1}{6^2}$ that the other two dice will both show the same number as the first one.) We found $p_X(3) = \frac{20}{36}$ in part (c) above. So $p_X(2) = 1 - \frac{1}{36} - \frac{20}{36} = \frac{15}{36} = \frac{5}{12}$.



Problem 2 *(16 points)* Let X be a random variable with mean 4 and variance 16. (a) Find the numerical value of $E[X^2]$. Remember to explain your reasoning. (b) Find the numerical value of $E[(X + 2)(X + 3)]$. Show your work.

Let σ_X^2 denote the variance of X and μ_X the mean of X. (a) Recall that $\sigma_X^2 = E[X^2] - \mu_X^2$, so that $E[X^2] = \mu_X^2 + \sigma_X^2 = 4^2 + 16 = 32$. (b) Multiplying out and using the linearity of expectation, we have
$E[((X + 2)(X + 3)] = E[X^2 + 5X + 6] = E[X^2] + 5E[X] + 6 = 32 + 5 \cdot 4 + 6 = 58$.

Problem 3 *(12 points)* Suppose that a random variable X has the pmf

$$p_X(k) = \begin{cases} (k-1)p^2(1-p)^{k-2} & k = 2, 3, \ldots \\ 0 & \text{else} \end{cases}$$

where p is an unknown parameter with $0 < p < 1$. (i.e., X has the negative binomial distribution with parameters p and $r = 2$.) Suppose it is observed that $X = 14$. What is the maximum likelihood estimate of p?

Setting $k = 14$, and dropping the factor $k-1$, we see we need to maximize $p^2(1-p)^{k-2}$ with respect to p, over the interval $[0,1]$. This function is zero at the endpoints 0 and 1, and is strictly positive inside the interval, so that the maximum does not occur at the endpoints.

$$\begin{aligned}(p^2(1-p)^{k-2})' &= 2p(1-p)^{k-2} - p^2(k-2)(1-p)^{k-3} \\ &= (2(1-p) - (k-2)p)\, p(1-p)^{k-3} \\ &= (2-kp)p(1-p)^{k-3}\end{aligned}$$

Thus, $\widehat{p} = \frac{2}{k} = \frac{1}{7}$. (Note: This result can also be deduced from the interpretation of the negative binomial distribution and the fact $\widehat{p} = \frac{k}{n}$ for estimation of the parameter p for a binomial distribution with known parameter n and unknown parameter p.)

Problem 4 *(14 points)* Let A and B denote events such that $P(A) = 0.6, P(B) = 0.5$, and $P(A\,|\,B) = 0.4$. (a) Find $P(A \cup B)$. (b) Find $P(B^c\,|\,A^c)$.

$P(AB) = P(A\,|\,B)P(B) = 0.2$, and from that fact it is easy to fill in the probabilities in the Karnaugh diagram:

	B^c	B
A^c	0.1	0.3
A	0.4	0.2

From that we see $P(A \cup B) = .4 + .2 + .3 = .9$, and $P(B^c|A^c) = \frac{P(A^cB^c)}{P(A^c)} = \frac{.1}{.1+.3} = \frac{1}{4}$.

Problem 5 *(22 points)* Consider repeated independent tosses of a biased coin with $P(\text{Heads}) = p$, and let X denote the number of tosses required to observe both one Head and one Tail. (a) What is the minimum possible value of X? (b) Find the probability mass function of X. (c) Find the expected value of X. (Find a closed form answer, with no infinite sum.)

(a) The minimum value of X is 2.

(b) For any k, $k \geq 2$, the event $\{X = k\}$ occurs if and only if the outcomes of the tosses are $HHH \cdots HT$ or $TTT \cdots TH$. Hence, with q denoting $1-p$, we have

$$p_X(k) = P\{X = k\} = p^{k-1}q + q^{k-1}p \text{ for } k \geq 2.$$

(c) Conditioning on the result of the first trial, we have $E[X|H] = 1 + \dfrac{1}{q}$, $\; E[X|T] = 1 + \dfrac{1}{p}$, since if the first trial resulted in a Head, we have to wait for an additional $\frac{1}{q}$ trials (on average) for a Tail, etc. Hence, $E[X] = \left(1 + \dfrac{1}{q}\right)p + \left(1 + \dfrac{1}{p}\right)q = \dfrac{1-p+p^2}{p(1-p)}$.

Alternatively, $E[X] = \sum_{k=2}^{\infty} k[p^{k-1}q + q^{k-1}p] = [2pq + 3p^2q + 4p^3q + \cdots] + [2qp + 3q^2p + 4q^3p + \cdots]$
$= [-q + \{1q + 2pq + 3p^2q + \cdots\}] + [-p + \{1p + 2qp + 3q^2p + \cdots\}]$
$= -q + \dfrac{1}{q} - p + \dfrac{1}{p} = \dfrac{1}{p} + \dfrac{1}{q} - 1 = \dfrac{1-p+p^2}{p(1-p)}$ where we have used the fact that the sums in curly brackets are the ones that occur in the calculation of the means of geometric random variables with parameters q and p respectively.