

1a) No, since it takes negative values, the value at $x=-2$ is not 0, and the value at $x=2$ is not 1.

1b) No.

$$f_X(x) \geq 0$$

$$Area = \int_{-\infty}^{\infty} \frac{1}{1+e^{-x}} dx = \int_{-\infty}^{\infty} \frac{e^x dx}{e^x + 1} = \int_1^{\infty} \frac{du}{u} = [\ln u]_1^{\infty} \neq 1$$

1c)

$$f_X(x) \geq 0$$
$$Area = 2 \int_0^{\infty} \frac{e^{-\alpha x}}{\alpha} dx = \frac{2e^{-\alpha x}}{-\alpha^2} \Big|_0^{\infty} = \frac{2}{\alpha^2}$$

Valid only if $\alpha = +\sqrt{2}$.

1d) No, because $F_X(x)$ has two values at $x = 0$!

2.

(a) Given an observation y , the ML decision rule is to choose $+d$ as the guess for the input if and only if

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-d)^2}{2\sigma^2}\right) \geq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y+d)^2}{2\sigma^2}\right)$$

which simplifies to choosing $+d$ as the guess for the input if and only if $y \geq 0$.

(b) The ML rule is in error when either one of the following occurs: $X = +d$ and $y < 0$, or $X = -d$ and $y \geq 0$.

$$\begin{aligned} P[\text{error}] &= P[y < 0 | X = +d] P[X = +d] + P[y \geq 0 | X = -d] P[X = -d] \\ &= P[N < -d] p + P[N > d] (1 - p) \\ &= P[N > d] = Q\left(\frac{d}{\sigma}\right) \end{aligned}$$

(c) The MAP rule is to choose $+d$ as the guess for the input if and only if

$$p \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-d)^2}{2\sigma^2}\right) \geq (1-p) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y+d)^2}{2\sigma^2}\right)$$

which reduces to choosing $+d$ as the guess for the input if and only if $y \geq \frac{\sigma^2}{2d} \log\left(\frac{1-p}{p}\right)$

(d) As long as $p < 1$, $\frac{\sigma^2}{2d} \log\left(\frac{1-p}{p}\right) > -\infty$ and hence the MAP rule will not be one that always picks $X = +d$ as the guess. Hence the answer is: only when $p = 1$.

3a)

$$\begin{aligned}\text{Area} &= \frac{1}{2}2ab + \frac{1}{2}(b-a)b \\ &= ab + \frac{b^2}{2} - \frac{ab}{2} \\ &= 1\end{aligned}$$

$$\frac{b^2}{2} + \frac{ab}{2} - 1 = 0$$

$$b^2 + ab - 2 = 0$$

3b)

$$a = \frac{2-b^2}{b} > 0 \implies b < \sqrt{2}$$

3c) If $b = 1$, then

$$a = \frac{2-1}{1} = 1$$

$$E(x) = \int_{-a}^a x \frac{b}{2a} (x+a) dx + 0$$

$$= \int_{-1}^1 x \frac{1}{2} (x+1) dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^1$$

$$= \frac{1}{2} \left[\frac{1}{3} + \frac{1}{2} + \frac{1}{3} - \frac{1}{2} \right]$$

$$= \frac{1}{3}$$

4a)

$$E(W) = E(X) - 5 = 0 \quad \text{so} \quad E(x) = 5$$

$$E(Z) = E(Y) + 2 = -3 \quad \text{so} \quad E(Y) = -5$$

$$E(Y) = aE(X) + b$$

$$-5 = a \cdot 5 + b \implies 5a + b = -5$$

$$1 = a \cdot 2 + b \implies 2a + b = 1$$

From the last two equations:

$$3a = -6 \implies a = -2$$

$$b = 1 - 2a = 1 + 4 = 5$$

4b)

$$\sigma_Y = |a|\sigma_X = 2 \cdot 3 = 6$$

4c)

$$\begin{aligned} E(W^2) &= \sigma_W^2 + [E(W)]^2 \\ &= \sigma_X^2 + 0^2 = 9 \end{aligned}$$

4d)

$$f_Y(ar + b) = \left|\frac{1}{a}\right| f_X(r) = \frac{1}{2} f_X(r)$$

5a)

$$f_T(t) = \lambda e^{-\lambda t} = \frac{e^{-t/4}}{4}$$
$$\lambda = .25$$
$$P(T \geq 8) = 1 - P(T < 8) = e^{-(.25)8} = e^{-2}$$

5b)

$$\sigma^2 = \frac{1}{\lambda^2} \implies \sigma = \frac{1}{\lambda} = 4$$
$$\mu = \frac{1}{\lambda} = 4$$
$$P(T \leq 2) + P(T \geq 6) = P(|t - 4| \geq 2)$$
$$= P[|t - \mu| \geq \frac{1}{2}\sigma] \leq \frac{1}{(1/2)^2} = 4 = P_{\max}$$

5c) A bound of 4 on probability is not very useful as probability is anyway ≤ 1 .

5d)

$$\sigma_5 = \sqrt{\frac{5}{\lambda^2}} = 4\sqrt{5}$$

6a)

$$\frac{\phi(-\sqrt{5})}{1 - \phi(+\sqrt{5})}$$

6b) Same as 6a.

6c)

$$\begin{aligned} & \phi\left(\frac{7}{\sqrt{5}}\right) - \phi\left(-\frac{3}{\sqrt{5}}\right) \\ &= 1 - P(m < 22) - P(m > 32) \\ &= 1 - [1 - \phi\left(\frac{3}{\sqrt{5}}\right)] - [1 - \phi\left(\frac{7}{\sqrt{5}}\right)] \\ &= \phi\left(\frac{3}{\sqrt{5}}\right) + \phi\left(\frac{7}{\sqrt{5}}\right) - 1 \end{aligned}$$

7a

(i) Both C and F do not fail with probability $(1 - p)^2$. In this case the two edges can be “shrunk” to form “intermediate nodes”. The conditional disconnection probability is thus

$$1 - (1 - p^2)^3$$

(ii) Both edges C and F fail with probability p^2 . When this happens there are two parallel paths from the source to the destination, and each has a probability $1 - (1 - p)^3$ of failing. Thus in this case the conditional disconnection probability is

$$(1 - (1 - p)^3)^2$$

(iii) Exactly one of the edges C and H fails with probability $2p(1 - p)$. In this case, shrink the un-failed edge. Now, there are two parallel length-two paths on one side of the intermediate node, and two parallel length-one paths on the other side. Thus, the conditional disconnection probability is

$$1 - (1 - p^2)(1 - (1 - (1 - p)^2)^2)$$

(b)

$$f(t) = \begin{cases} 0 & , \quad t \leq t_{safe} \\ \frac{e^{-(t-t_{safe})}}{400} & , \quad t > t_{safe} \end{cases}$$