

ECE 413: Problem Set 14

Due: Wednesday May 4 at the beginning of class.
Reading: Ross, Chapter 7, Sections 1-5, Chapter 8, Sections 1-4
Noncredit exercises: Ross Chapter 7: Problems 1, 16, 26, 30 33, 34, 38;
 Theoretical Exercises: 1, 2, 17, 22, 23, 40
 Chapter 8: problems 1-9, 15.

This Problem Set contains six problems

1. Ross Problem 43, p. 294.
2. (Unbelievable but true: this problem is *much* easier than it looks ...).
 - (a) If $\mathcal{X}$ is $N(0, \sigma^2)$, use the magic formula in Example 7b, Chapter 5.7 of Ross to show that $\mathcal{X}^2$ has *gamma* pdf with parameter $(\frac{1}{2}, \frac{1}{2\sigma^2})$.
 - (b) Now, suppose that $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ are *independent* $N(0, \sigma^2)$ random variables. Then $\mathcal{X}^2, \mathcal{Y}^2$, and $\mathcal{Z}^2$ are independent gamma random variables with parameter $(\frac{1}{2}, \frac{1}{2\sigma^2})$. Use the comment immediately following the proof of Proposition 3.1 on p. 262 of Ross to *state* what is the type of pdf of $\mathcal{W} = \mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2$, and *write down* explicitly the exact function $f_{\mathcal{W}}(\alpha)$. What is the numerical value of $f_{\mathcal{W}}(5)$ if $\sigma^2 = 4$?
 - (c) Prove that $E[\mathcal{W}] = 3\sigma^2$. If you actually evaluated an integral to get this answer instead of using LOTUS, shame on you!
 - (d) In a physical application, $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ represent the velocity (measured along three perpendicular axes) of a gas molecule of mass m . Thus, $\mathcal{H} = \frac{1}{2}m\mathcal{W}$ is the kinetic energy of the particle. It is an axiom of statistical mechanics that the *average* kinetic energy is $E[\mathcal{H}] = E[\frac{1}{2}m\mathcal{W}] = \frac{1}{2}mE[\mathcal{W}] = \frac{3}{2}m\sigma^2 = \frac{3}{2}kT$ where k is Boltzmann's constant and T is the absolute temperature of the gas in $^{\circ}K$. (Note that the average energy is $\frac{1}{2}kT$ per dimension.) Show that the kinetic energy $\mathcal{H}$ has the Maxwell-Boltzmann pdf: $f_{\mathcal{H}}(\beta) = \frac{2}{\sqrt{\pi}}(kT)^{-\frac{3}{2}}\sqrt{\beta}\exp\left(-\frac{\beta}{kT}\right)$ for $\beta \geq 0$.
 - (e) $\mathcal{V} = \sqrt{\mathcal{W}} = \sqrt{\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2}$ is the *speed* of the molecule. Show that the pdf of $\mathcal{V}$ is $f_{\mathcal{V}}(\gamma) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{\frac{3}{2}}\gamma^2\exp\left(-\frac{m\gamma^2}{2kT}\right)$, for $\gamma \geq 0$. cf. Theoretical Exercise 1 of Chapter 5.
 - (f) What is the average speed of the molecule?
3. Consider the random point $(\mathcal{X}, \mathcal{Y})$ whose joint pdf was specified in Problem 1 of Problem Set 13.
 - (a) Find $E[\mathcal{X}]$ and $\text{var}(\mathcal{X})$.
 - (b) Explain why the random variable $\mathcal{Y}$ has the same mean and variance as $\mathcal{X}$.
 - (c) Compute $E[\mathcal{X}\mathcal{Y}]$ and hence determine $\text{cov}(\mathcal{X}, \mathcal{Y})$.

4. Let $E[\mathcal{X}] = 1$, $E[\mathcal{Y}] = 4$, $\text{var}(\mathcal{X}) = 4$, $\text{var}(\mathcal{Y}) = 9$, and $\rho_{\mathcal{X},\mathcal{Y}} = 0.1$.
- (a) If $\mathcal{Z} = 2(\mathcal{X} + \mathcal{Y})(\mathcal{X} - \mathcal{Y})$, what is $E[\mathcal{Z}]$?
 - (b) If $\mathcal{T} = 2\mathcal{X} + \mathcal{Y}$ and $\mathcal{U} = 2\mathcal{X} - \mathcal{Y}$, what is $\text{cov}(\mathcal{T}, \mathcal{U})$?
 - (c) Find the mean and variance of $\mathcal{W} = 3\mathcal{X} + \mathcal{Y} + 2$.
 - (d) If $\mathcal{X}$ and $\mathcal{Y}$ are jointly Gaussian random variables, and $\mathcal{W}$ is as defined in part (c), what is $P\{\mathcal{W} > 0\}$?
5. This problem has three independent parts. Do not apply the numbers from one part to the others.
- (a) If $\text{var}(\mathcal{X} + \mathcal{Y}) = 36$ and $\text{var}(\mathcal{X} - \mathcal{Y}) = 64$, what is $\text{cov}(\mathcal{X}, \mathcal{Y})$? If you are also told that $\text{var}(\mathcal{X}) = 3 \cdot \text{var}(\mathcal{Y})$, what is $\rho_{\mathcal{X},\mathcal{Y}}$?
 - (b) If $\text{var}(\mathcal{X} + \mathcal{Y}) = \text{var}(\mathcal{X} - \mathcal{Y})$, are $\mathcal{X}$ and $\mathcal{Y}$ uncorrelated?
 - (c) If $\text{var}(\mathcal{X}) = \text{var}(\mathcal{Y})$, are $\mathcal{X}$ and $\mathcal{Y}$ uncorrelated?
6. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are zero-mean jointly Gaussian random variables with variances σ_1^2 and σ_2^2 respectively and correlation coefficient ρ .
- (a) Find the means and variances of the random variables $\mathcal{Z} = \mathcal{X} \cos \theta + \mathcal{Y} \sin \theta$ and $\mathcal{W} = \mathcal{Y} \cos \theta - \mathcal{X} \sin \theta$.
 - (b) What is $\text{cov}(\mathcal{Z}, \mathcal{W})$?
 - (c) Find an angle θ such that $\mathcal{Z}$ and $\mathcal{W}$ are *independent* Gaussian random variables. You may express your answer as a trigonometric function involving σ_1^2 , σ_2^2 , and ρ . In particular, what is the value of θ if $\sigma_1 = \sigma_2$?