

## ECE 413: Problem Set 12

**Due:** Wednesday April 20 at the beginning of class.

**Reading:** Ross, Chapter 6

**Noncredit exercises:** Ross, Chapter 6, Problems 1, 9-15, 20-23

**This Problem Set contains six problems**

1. The joint pmf  $p_{\mathcal{X},\mathcal{Y}}(u, v)$  of  $\mathcal{X}$  and  $\mathcal{Y}$  is shown in the table below.

|                                                              |      |      |      |      |
|--------------------------------------------------------------|------|------|------|------|
| $\begin{array}{c} u \rightarrow \\ v \downarrow \end{array}$ | 0    | 1    | 3    | 5    |
| 4                                                            | 0    | 1/12 | 1/6  | 1/12 |
| 3                                                            | 1/6  | 1/12 | 0    | 1/12 |
| -1                                                           | 1/12 | 1/6  | 1/12 | 0    |

- Find the marginal pmfs  $p_{\mathcal{X}}(u)$  and  $p_{\mathcal{Y}}(v)$ .
  - Are  $\mathcal{X}$  and  $\mathcal{Y}$  independent random variables?
  - Find  $P\{\mathcal{X} \leq \mathcal{Y}\}$  and  $P\{\mathcal{X} + \mathcal{Y} \leq 4\}$ .
  - Find  $p_{\mathcal{X}|\mathcal{Y}}(u|3)$ ,  $E[\mathcal{X}|\mathcal{Y} = 3]$  and  $\text{var}(\mathcal{X}|\mathcal{Y} = 3)$ .
2. Packets arriving at a router are addressed either to Server A or to Server B. Each packet address (A or B) may be regarded as an independent trial of an experiment whose outcomes are A and B with probabilities  $p$  and  $q = 1 - p$  respectively. The router sends each packet to the appropriate server.

We model the packet arrivals as a Poisson process with arrival rate  $\lambda$ . Let  $\mathcal{X} = N(0, T]$  denote the number of packets arriving at the router during  $(0, T]$  and  $\mathcal{Y} = N_A(0, T]$  the number of packets that are routed to Server A. Then, all the above implies that *given* that  $\mathcal{X} = n$ , the *conditional* pmf  $p_{\mathcal{Y}|\mathcal{X}}(m|n)$  of  $\mathcal{Y}$  is binomial with parameters  $(n, p)$ . Of course,  $\mathcal{X}$  is a Poisson random variable with parameter  $\lambda T$ , and thus, as discussed in class, the *unconditional* pmf of  $\mathcal{Y}$  is Poisson with parameter  $(\lambda p)T$ .

- What is the conditional pmf  $p_{\mathcal{X}|\mathcal{Y}}(n|m)$  of  $\mathcal{X}$  given that  $\mathcal{Y}$  was observed to have value  $m$ ?
  - What is  $E[\mathcal{X}|\mathcal{Y} = m]$ , the conditional expectation of  $\mathcal{X}$  given that  $\mathcal{Y}$  was observed to have value  $m$ ?
3. The jointly continuous random variables  $\mathcal{X}$  and  $\mathcal{Y}$  have joint pdf

$$f_{\mathcal{X},\mathcal{Y}}(u, v) = \begin{cases} 0.5, & 0 \leq u < 1, 0 \leq v < 1, \text{ and } 0 \leq u + v < 1, \\ 1.5, & 0 \leq u < 1, 0 \leq v < 1, \text{ and } 1 \leq u + v < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

- Find the marginal pdf of  $\mathcal{X}$ .
- Find  $P\{\mathcal{X} + \mathcal{Y} \leq 3/2\}$  and  $P\{\mathcal{X}^2 + \mathcal{Y}^2 \geq 1\}$ .

4. Ross, Problem 8, page 290.

5. The jointly continuous random variables  $\mathcal{X}$  and  $\mathcal{Y}$  have joint pdf

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} 2\exp(-u-v), & 0 < u < v < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Sketch the  $u$ - $v$  plane and indicate on it the region over which  $f_{\mathcal{X},\mathcal{Y}}(u,v)$  is nonzero.
  - (b) Find the marginal pdfs of  $\mathcal{X}$  and  $\mathcal{Y}$ .
  - (c) Are the random variables  $\mathcal{X}$  and  $\mathcal{Y}$  independent ?
  - (d) Find  $P\{\mathcal{Y} > 3\mathcal{X}\}$ .
  - (e) For  $\alpha > 0$ , find  $P\{\mathcal{X} + \mathcal{Y} \leq \alpha\}$ .
  - (f) Use the result in part (e) to determine the pdf of the random variable  $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$ .
6. We return to the “random chord” of Problem 2 of Problem Set 11. Yet another way of defining a “random chord” is to choose the midpoint of the chord to be anywhere inside the circle of radius 1 with equal probability. The chord is, of course, perpendicular to the diameter that passes through the chosen point. Thus, let the random point  $(\mathcal{X}, \mathcal{Y})$  be *uniformly distributed* on the interior of the circle of unit radius centered at the origin (this region is called the unit disc — nomenclature that might be familiar to DSPists).
- (a) Find the probability that the length  $\mathcal{L}$  of the random chord is greater than the side of the equilateral triangle inscribed in the circle.
  - (b) Express  $\mathcal{L}$  as a function of the random variable  $(\mathcal{X}, \mathcal{Y})$  and find the probability density function for  $\mathcal{L}$ .
  - (c) Find the average length of the chord, i.e. find  $E[\mathcal{L}]$ .