

ECE 413: Solutions to Problem Set 14

1. Conditioned on $\mathcal{X} = u$, $\mathcal{Y}$ takes on values in $(-u, u)$. Now, from Problem 4 on Problem Set 12, we know that $f_{\mathcal{X}}(u) = (1/6)u^3 \exp(-u)$ for $u \geq 0$. (We do? How and why? Isn't it bass backwards?)

$$\text{Hence, } f_{\mathcal{Y}|\mathcal{X}}(v|u) = \frac{f_{\mathcal{X},\mathcal{Y}}(u,v)}{f_{\mathcal{X}}(u)} = \frac{(1/8)(u^2 - v^2) \exp(-u)}{(1/6)u^3 \exp(-u)} = \frac{3}{4u} \left[1 - \frac{v^2}{u^2} \right] \text{ for } -u \leq v \leq u.$$

$$\begin{aligned} \text{2. (a) } f_{\mathcal{X}^2}(v) &= \frac{1}{2\sqrt{v}} [f_{\mathcal{X}}(\sqrt{v}) + f_{\mathcal{X}}(-\sqrt{v})] = \frac{1}{2\sqrt{v}} \times \frac{1}{\sigma\sqrt{2\pi}} [\exp(-v/2\sigma^2) + \exp(-v/2\sigma^2)] \\ &= \frac{1}{\sigma\sqrt{2v}} \times \frac{1}{\sqrt{\pi}} \exp\left(-\frac{v}{2\sigma^2}\right) = \frac{\lambda(\lambda v)^{\frac{1}{2}-1}}{\Gamma\left(\frac{1}{2}\right)} \exp(-\lambda v) \text{ where } \lambda = \frac{1}{2\sigma^2} \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \end{aligned}$$

This is a *gamma* pdf with parameters $\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$.

- (b) The sum of independent gamma random variables with parameters (t_i, λ) is a gamma random variable with parameters $(\sum t_i, \lambda)$. Hence, $\mathcal{W} = \mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2$ is a gamma random variable with parameters $\left(\frac{3}{2}, \frac{1}{2\sigma^2}\right)$ whose pdf is $f_{\mathcal{W}}(\alpha) = \begin{cases} \frac{1}{\sigma^3} \sqrt{\frac{\alpha}{2\pi}} \exp\left(-\frac{\alpha}{2\sigma^2}\right), & \alpha \geq 0, \\ 0, & \alpha < 0. \end{cases}$

If $\sigma^2 = 4$, $f_{\mathcal{W}}(5) = (1/8)\sqrt{5/2\pi} \exp(-5/8)$.

- (c) $\mathbb{E}[\mathcal{W}] = \mathbb{E}[\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2] = \mathbb{E}[\mathcal{X}^2] + \mathbb{E}[\mathcal{Y}^2] + \mathbb{E}[\mathcal{Z}^2] = 3\sigma^2$ since $\mathcal{W}$ is a gamma random variable (with parameters $(3/2, 1/2\sigma^2)$), and its expected value is the ratio of the parameters, *viz.* $3\sigma^2$.
- (d) The pdf of $\mathcal{H} = \frac{1}{2}m\mathcal{W}$ is $f_{\mathcal{H}}(\beta) = (2/m)f_{\mathcal{W}}(2\beta/m)$. Since $\sigma^2 = \frac{kT}{m}$, we get that the kinetic energy $\mathcal{H}$ has the Maxwell-Boltzmann pdf: $f_{\mathcal{H}}(\beta) = \frac{2}{\sqrt{\pi}}(kT)^{-\frac{3}{2}}\sqrt{\beta} \exp\left(-\frac{\beta}{kT}\right)$ for $\beta \geq 0$.
- (e) $F_{\mathcal{V}}(\gamma) = P\{\mathcal{V} \leq \gamma\} = P\{\mathcal{W} \leq \gamma^2\} = F_{\mathcal{W}}(\gamma^2)$. Hence,

$$f_{\mathcal{V}}(\gamma) = 2\gamma f_{\mathcal{W}}(\gamma^2) = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT}\right)^{\frac{3}{2}} \gamma^2 \exp\left(-\frac{m\gamma^2}{2kT}\right) \text{ for } \gamma \geq 0.$$

$$\begin{aligned} \text{(f) } \mathbb{E}[\mathcal{V}] &= \int_0^\infty \gamma \cdot \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT}\right)^{\frac{3}{2}} \gamma^2 \exp\left(-\frac{m\gamma^2}{2kT}\right) d\gamma = \int_0^\infty 2\sqrt{\frac{2kT}{m\pi}} x \cdot \exp(-x) dx = 2\sqrt{\frac{2kT}{m\pi}} \text{ on} \\ &\text{substituting } m\gamma^2/2kT = x. \text{ Alternatively,} \\ \mathbb{E}[\mathcal{V}] &= \mathbb{E}[\sqrt{\mathcal{W}}] = \int_0^\infty \sqrt{\alpha} \frac{1}{\sigma^3} \sqrt{\frac{\alpha}{2\pi}} \exp\left(-\frac{\alpha}{2\sigma^2}\right) d\alpha = \int_0^\infty \frac{4\sigma}{\sqrt{2\pi}} x \cdot \exp(-x) dx = \frac{4\sigma}{\sqrt{2\pi}} = 2\sqrt{\frac{2kT}{m\pi}} \\ &\text{on substituting } \alpha/2\sigma^2 = x \text{ and remembering that } \sigma^2 = \frac{kT}{m}. \end{aligned}$$

3. The marginal pdf of $\mathcal{X}$ has value $f_{\mathcal{X}}(u) = \begin{cases} \frac{2}{3}, & 0 \leq u \leq \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$

$$\text{(a) } \mathbb{E}[\mathcal{X}] = \int_0^{1/2} u \cdot \frac{2}{3} du + \int_{1/2}^1 u \cdot \frac{4}{3} du = \frac{u^2}{3} \Big|_0^{1/2} + \frac{2u^2}{3} \Big|_{1/2}^1 = \frac{1}{12} + \frac{2}{3} - \frac{1}{6} = \frac{7}{12}.$$

Alternatively, we know from Problem Set 13 that $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha)$ is $U[1/2, 1]$ or $U[0, 1]$ according as $\mathcal{Y} < 1/2$ or $\mathcal{Y} > 1/2$. The corresponding conditional means are $3/4$ and $1/2$. Hence,

$$\mathbb{E}[\mathcal{X}] = \frac{3}{4} \times P\{\mathcal{Y} < 1/2\} + \frac{1}{2} \times P\{\mathcal{Y} > 1/2\} = \frac{3}{4} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{7}{12}.$$

$$\mathbb{E}[\mathcal{X}^2] = \int_0^{1/2} u^2 \cdot \frac{2}{3} du + \int_{1/2}^1 u^2 \cdot \frac{4}{3} du = \frac{2u^3}{9} \Big|_0^{1/2} + \frac{4u^3}{9} \Big|_{1/2}^1 = \frac{1}{36} + \frac{4}{9} - \frac{1}{18} = \frac{15}{36} = \frac{5}{12}.$$

$$\text{var}(\mathcal{X}) = \mathbb{E}[\mathcal{X}^2] - (\mathbb{E}[\mathcal{X}])^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{60}{144} - \frac{49}{144} = \frac{11}{144}.$$

Alternatively, the conditional variance of $\mathcal{X}$ is $1/48$ or $1/12$ according as $\mathcal{Y} < 1/2$ or $\mathcal{Y} > 1/2$.
The mean of the conditional variance is $(1/48) \times (1/3) + (1/12) \times (2/3) = (1/144) + (8/144) = 9/144$.
The variance of the conditional mean is $(3/4)^2(1/3) + (1/2)^2(2/3) - (7/12)^2$
 $= (3/16) + (1/6) - (49/144) = (27 + 24 - 49)/(144) = 2/144$.
The sum of these two quantities is $\text{var}(\mathcal{X}) = 11/144$ as before.

(b) Since $\mathcal{Y}$ has the same pdf as $\mathcal{X}$, it has the same mean and variance as $\mathcal{X}$.

$$(c) \quad E[\mathcal{X}\mathcal{Y}] = \int_0^1 \int_0^1 uv \frac{4}{3} du dv - \int_0^{1/2} \int_0^{1/2} uv \frac{4}{3} du dv = \frac{4}{3} \int_0^1 u du \int_0^1 v dv - \frac{4}{3} \int_0^{1/2} u du \int_0^{1/2} v dv$$

$$= \frac{4}{3} \left[\frac{1}{2} \times \frac{1}{2} - \frac{1}{8} \times \frac{1}{8} \right] = \frac{5}{16}. \text{ Hence, } \text{cov}(\mathcal{X}, \mathcal{Y}) = E[\mathcal{X}\mathcal{Y}] - E[\mathcal{X}]E[\mathcal{Y}] = \frac{5}{16} - \frac{49}{144} = -\frac{4}{144} = -\frac{1}{36}.$$

4. $E[\mathcal{X}] = 1, E[\mathcal{Y}] = 4, \text{var}(\mathcal{X}) = 4, \text{var}(\mathcal{Y}) = 9$, and $\rho_{\mathcal{X}, \mathcal{Y}} = 0.1$.

- (a) $E[\mathcal{Z}] = E[2(\mathcal{X} + \mathcal{Y})(\mathcal{X} - \mathcal{Y})] = 2E[\mathcal{X}^2 - \mathcal{Y}^2] = 2E[\mathcal{X}^2] - 2E[\mathcal{Y}^2] = 2[4 + 1^2] - 2[9 + 4^2] = -40$.
(b) $\text{cov}(\mathcal{T}, \mathcal{U}) = \text{cov}(2\mathcal{X} + \mathcal{Y}, 2\mathcal{X} - \mathcal{Y}) = 4 \cdot \text{cov}(\mathcal{X}, \mathcal{X}) + 2 \cdot \text{cov}(\mathcal{Y}, \mathcal{X}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) - \text{cov}(\mathcal{Y}, \mathcal{Y})$
 $= 4 \cdot \text{var}(\mathcal{X}) + 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) - \text{var}(\mathcal{Y}) = 4 \cdot \text{var}(\mathcal{X}) - \text{var}(\mathcal{Y}) = 4 \cdot 4 - 9 = 7$.
(c) $E[\mathcal{W}] = E[3\mathcal{X} + \mathcal{Y} + 2] = 3E[\mathcal{X}] + E[\mathcal{Y}] + 2 = 9$.
 $\text{var}(\mathcal{W}) = \text{var}(3\mathcal{X} + \mathcal{Y} + 2) = 3^2 \cdot \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) + 2 \cdot 3 \cdot 1 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) = 9 \cdot 4 + 9 + 6 \cdot 2 \cdot 3 \cdot 0.1 = 48.6$.
(d) $P\{\mathcal{W} > 0\} = 1 - \Phi\left(\frac{0-9}{\sqrt{48.6}}\right) = 1 - \Phi\left(-\frac{9}{\sqrt{48.6}}\right) = \Phi\left(\frac{9}{\sqrt{48.6}}\right)$.

5. (a) $\text{var}(\mathcal{X} + \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) + 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) = 36$.
 $\text{var}(\mathcal{X} - \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y}) = 64$. Hence, $\text{cov}(\mathcal{X}, \mathcal{Y}) = -7$.
From the above, $2 \cdot \text{var}(\mathcal{X}) + 2 \cdot \text{var}(\mathcal{Y}) = 8 \cdot \text{var}(\mathcal{Y}) = 100$, giving $\text{var}(\mathcal{Y}) = 12.5, \text{var}(\mathcal{X}) = 37.5$ and
 $\rho_{\mathcal{X}, \mathcal{Y}} = \text{cov}(\mathcal{X}, \mathcal{Y}) / \sqrt{\text{var}(\mathcal{X})\text{var}(\mathcal{Y})} = -7/12.5\sqrt{3}$.
(b) $\text{var}(\mathcal{X} + \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) + 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y})$ equals $\text{var}(\mathcal{X} - \mathcal{Y}) = \text{var}(\mathcal{X}) + \text{var}(\mathcal{Y}) - 2 \cdot \text{cov}(\mathcal{X}, \mathcal{Y})$
if and only if $\text{cov}(\mathcal{X}, \mathcal{Y}) = 0$, that is, if and only if $\mathcal{X}$ and $\mathcal{Y}$ are uncorrelated.
(c) No, whether $\text{var}(\mathcal{X})$ equals $\text{var}(\mathcal{Y})$ or not has no bearing on the question of whether $\text{cov}(\mathcal{X}, \mathcal{Y})$ is zero or not.

6. This problem considers the following situation. A random point in the plane has coordinates $(\mathcal{X}, \mathcal{Y})$ with respect to our chosen axes, and these coordinates are jointly Gaussian random variables. If we rotate the axes by an angle θ , the coordinates of the same random point *with respect to the new axes* are $(\mathcal{Z}, \mathcal{W})$. The joint pdf surface is still the same, but the coordinate axes are different, and hence the mathematical formula expressing the value of the joint pdf is different: in fact, it is a jointly Gaussian pdf with means, variances, and covariance given by the answers to be found in parts (a) and (b).

Note that $E[\mathcal{X}] = E[\mathcal{Y}] = 0$ and hence $E[\mathcal{X}^2] = \sigma_1^2$ and $E[\mathcal{Y}^2] = \sigma_2^2$ while $\text{cov}[\mathcal{X}, \mathcal{Y}] = \rho \cdot \sigma_1 \sigma_2$.

- (a) $E[\mathcal{Z}] = E[\mathcal{X} \cos \theta + \mathcal{Y} \sin \theta] = E[\mathcal{X}] \cos \theta + E[\mathcal{Y}] \sin \theta = 0$.
 $E[\mathcal{W}] = E[\mathcal{Y} \cos \theta - \mathcal{X} \sin \theta] = 0$ also.
Since $\mathcal{Z}$ and $\mathcal{W}$ are zero-mean random variables, we get
 $\text{var}(\mathcal{Z}) = E[\mathcal{Z}^2] = E[\mathcal{X}^2 \cos^2 \theta + \mathcal{Y}^2 \sin^2 \theta + 2\mathcal{X}\mathcal{Y} \sin \theta \cos \theta] = \sigma_1^2 \cos^2 \theta + \sigma_2^2 \sin^2 \theta + 2\rho\sigma_1\sigma_2 \sin \theta \cos \theta$,
and
 $\text{var}(\mathcal{W}) = E[\mathcal{W}^2] = E[\mathcal{X}^2 \sin^2 \theta + \mathcal{Y}^2 \cos^2 \theta - 2\mathcal{X}\mathcal{Y} \sin \theta \cos \theta] = \sigma_1^2 \sin^2 \theta + \sigma_2^2 \cos^2 \theta - 2\rho\sigma_1\sigma_2 \sin \theta \cos \theta$.

(b) Since $\mathcal{Z}$ and $\mathcal{W}$ are zero-mean random variables,

$$\begin{aligned} \text{cov}(\mathcal{Z}, \mathcal{W}) = E[\mathcal{Z}\mathcal{W}] &= E[(\mathcal{X} \cos \theta + \mathcal{Y} \sin \theta)(\mathcal{Y} \cos \theta - \mathcal{X} \sin \theta)] \\ &= E[\mathcal{Y}^2] \sin \theta \cos \theta - E[\mathcal{X}^2] \sin \theta \cos \theta + E[\mathcal{X}\mathcal{Y}](\cos^2 \theta - \sin^2 \theta) \\ &= \sin \theta \cos \theta \cdot (\sigma_2^2 - \sigma_1^2) + (\cos^2 \theta - \sin^2 \theta) \cdot \rho\sigma_1\sigma_2 \\ &= \frac{1}{2} \cdot \sin 2\theta \cdot (\sigma_2^2 - \sigma_1^2) + \cos 2\theta \cdot \rho\sigma_1\sigma_2 \end{aligned}$$

(c) $\mathcal{Z}$ and $\mathcal{W}$ are jointly Gaussian random variables and thus they are independent if $\text{cov}(\mathcal{Z}, \mathcal{W}) = 0$.

We get independent random variables if we choose $\theta = \frac{1}{2} \arctan\left(\frac{2\rho \cdot \sigma_1 \sigma_2}{\sigma_1^2 - \sigma_2^2}\right)$. Note that if θ_0 is a solution to this equation, then $\theta_0 + \pi$ is also a solution, as are $\theta_0 \pm \pi/2$. That is, there are *four* different values of θ in the range $[0, 2\pi)$ that can be used to get independent Gaussian random variables from $\mathcal{X}$ and $\mathcal{Y}$. In particular, if $\sigma_1 = \sigma_2$, then θ can take on values $\pm\pi/4$ and $\pm3\pi/4$.