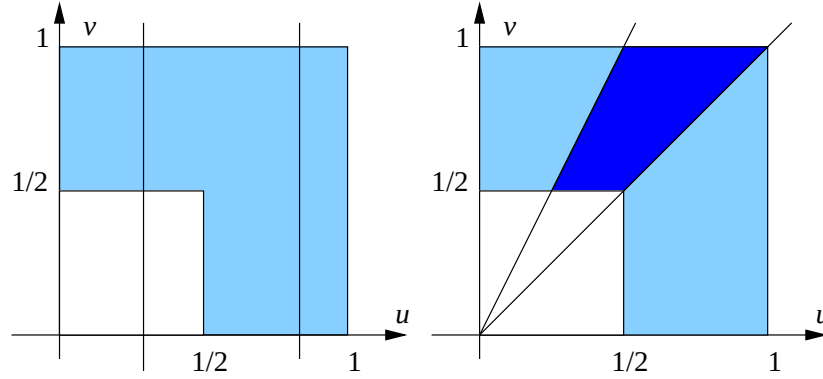


ECE 413: Solutions to Problem Set 13

1. The random point $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the shaded region shown in the figures below. Clearly, $f_{\mathcal{X}, \mathcal{Y}}(u, v) = \frac{4}{3}$ on this region.



- (a) $f_{\mathcal{X}}(u)$ is the area of the cross-section of the pdf surface along the line u . There are two cases to be considered, as shown in the left-hand figure above. It is obvious almost by inspection that

$$f_{\mathcal{X}}(u) = \begin{cases} \frac{2}{3}, & 0 \leq u \leq \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- (b) By symmetry, $f_{\mathcal{X}}$ and $f_{\mathcal{Y}}$ are the same function: $f_{\mathcal{Y}}(v) = \begin{cases} \frac{2}{3}, & 0 \leq v \leq \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} < v \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$

- (c) $P\{\mathcal{X} < \mathcal{Y} < 2\mathcal{X}\}$ is the volume in the deep-shaded trapezoidal region in the right-hand figure above. The area of the region is $\frac{1}{2} \left[\frac{1}{2} + \frac{1}{4} \right] \frac{1}{2} = \frac{3}{16}$ and hence $P\{\mathcal{X} < \mathcal{Y} < 2\mathcal{X}\} = \frac{3}{16} \times \frac{4}{3} = \frac{1}{4}$.

- (d) Given that $\mathcal{Y} = \alpha$, the conditional pdf $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha)$ is the cross-section of the pdf surface at $v = \alpha$ unitized to have area 1.

Thus, for $0 < \alpha < 1/2$, $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) \sim U[1/2, 1]$, while if $1/2 < \alpha < 1$, $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) \sim U[0, 1]$.

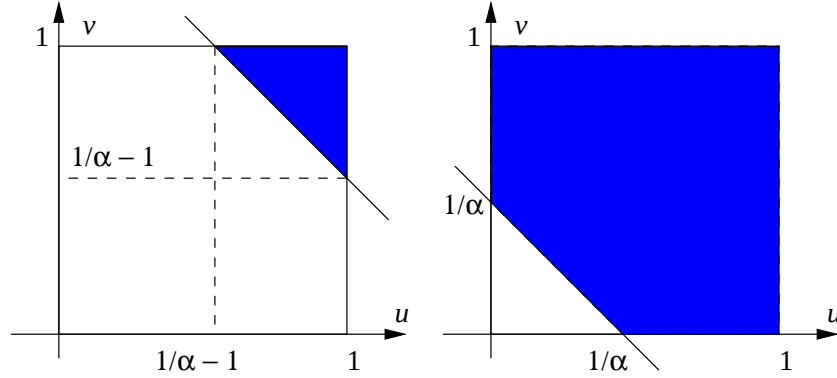
- (e) $f_{\mathcal{X}}(u) = \int_0^1 f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) f_{\mathcal{Y}}(\alpha) d\alpha$.

- For fixed u , $0 < u < 1/2$, $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) = 0$ and $f_{\mathcal{Y}}(\alpha) = \frac{2}{3}$ if $0 < \alpha < 1/2$. Also, $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) = 1$ and $f_{\mathcal{Y}}(\alpha) = \frac{4}{3}$ if $1/2 < \alpha < 1$. Hence, $f_{\mathcal{X}}(u) = 1 \times \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$ for $0 < u < 1/2$.
- for fixed u , $1/2 < u < 1$, $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) = 2$ and $f_{\mathcal{Y}}(\alpha) = \frac{2}{3}$ if $0 < \alpha < 1/2$. Also, $f_{\mathcal{X}|\mathcal{Y}}(u|\alpha) = 1$ and $f_{\mathcal{Y}}(\alpha) = \frac{4}{3}$ if $1/2 < \alpha < 1$. Hence, $f_{\mathcal{X}}(u) = 2 \times \frac{1}{2} \times \frac{2}{3} + 1 \times \frac{1}{2} \times \frac{4}{3} = \frac{4}{3}$ for $1/2 < u < 1$.

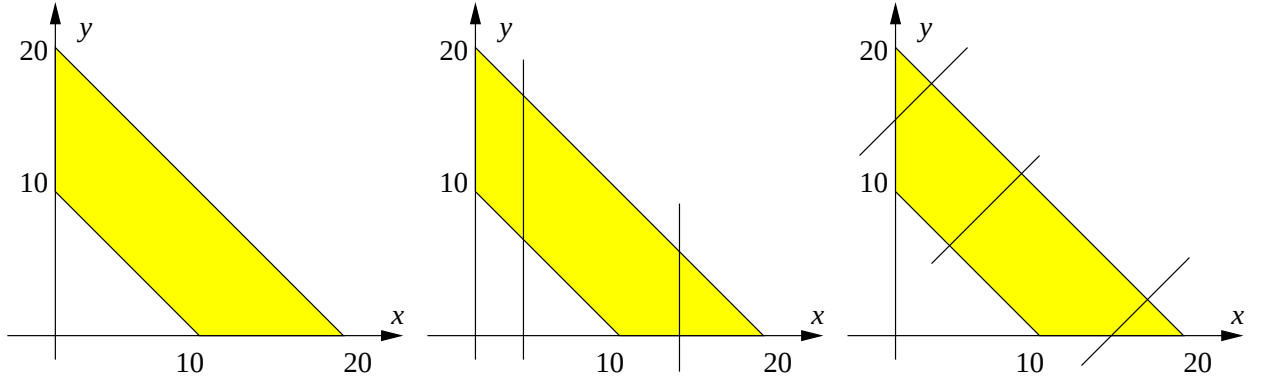
This is the same answer as in part (a).

2. $\mathcal{I} = 1/(\mathcal{R}_1 + \mathcal{R}_2)$ takes on values in $(1/2, \infty)$. Now, $F_{\mathcal{I}}(\alpha) = P\{\mathcal{I} \leq \alpha\} = P\{\mathcal{R}_1 + \mathcal{R}_2 \geq \alpha^{-1}\}$. For $1/2 \leq \alpha \leq 1$, we see from the left-hand figure below that $F_{\mathcal{I}}(\alpha) = \frac{1}{2}(1 - (\alpha^{-1} - 1))^2 = \frac{1}{2}(2 - \alpha^{-1})^2$. For $\alpha > 1$, we see from the right-hand figure below that $F_{\mathcal{I}}(\alpha) = 1 - \frac{1}{2}(\alpha^{-1})^2$.

Consequently, $f_{\mathcal{I}}(\alpha) = \begin{cases} 2\alpha^{-2} - \alpha^{-3}, & \frac{1}{2} \leq \alpha \leq 1, \\ \alpha^{-3}, & \alpha > 1, \\ 0, & \text{elsewhere.} \end{cases}$



3. $f_{\mathcal{R}, \mathcal{S}}(x, y) = K$ on the shaded region shown in the left-hand figure below.



- (a) The area of the trapezoid is $\frac{1}{2}(20)^2 - 10^2 = 150$, and hence $K = \frac{1}{150}$.

- (b) $f_{\mathcal{R}}(x)$, the marginal pdf of $\mathcal{R}$, is the area of the cross-section of the joint pdf surface at x . The cross-section is always a rectangle of height $\frac{1}{150}$ but the base is 10 if $0 \leq x \leq 10$, while the base is $(20 - x)$ if $10 < x \leq 20$, as indicated in the middle figure above.

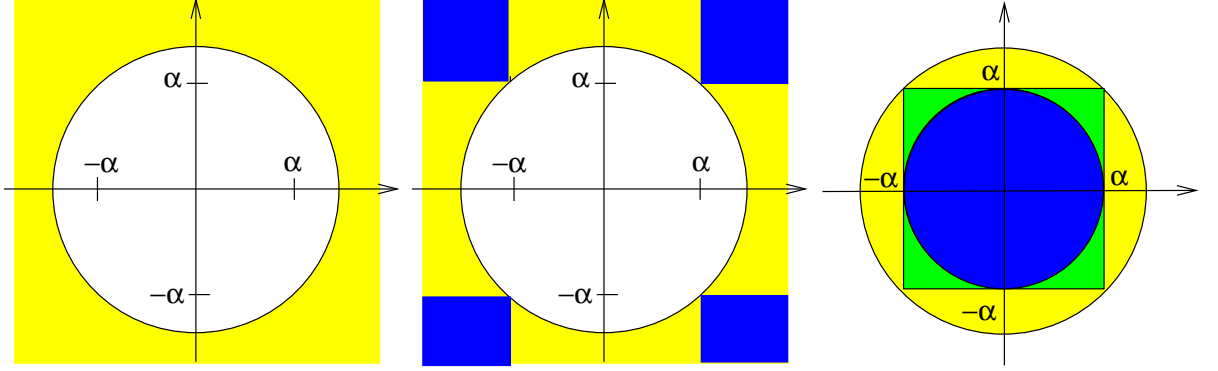
Hence, $f_{\mathcal{R}}(x) = \begin{cases} \frac{1}{15}, & 0 \leq x \leq 10, \\ \frac{1}{15} - \frac{x}{150}, & 10 < x \leq 20, \\ 0, & \text{elsewhere.} \end{cases}$ Note that $P\{\mathcal{R} \leq 10\} = \frac{2}{3}$.

- (c) The sleepless student pulling an all-nighter ($\mathcal{R} = 20$ and $\mathcal{S} = 0$) achieves $\mathcal{T} = 0$ while the well-rested student who has worked diligently on ECE 413 all semester but not specifically for the exam ($\mathcal{R} = 0$ and $\mathcal{S} = 20$) acs the exam! Now, $F_{\mathcal{T}}(\alpha) = P\{\mathcal{T} \leq \alpha\} = P\{50 + 2.5(\mathcal{S} - \mathcal{R}) \leq \alpha\} = P\{\mathcal{S} \leq \mathcal{R} + 0.4\alpha - 20\}$ is the probability mass below the line of unit slope and intercept $20 - 0.4\alpha$ on the x axis. There are three special cases to be considered as indicated in the right-hand figure above:

- For $0 \leq \alpha \leq 25$, $F_{\mathcal{T}}(\alpha) = \frac{1}{4}(0.4\alpha)^2 \frac{1}{150}$ and therefore $f_{\mathcal{T}}(\alpha) = \frac{\alpha}{1875}$. Note that $F_{\mathcal{T}}(25) = \frac{1}{6}$.
- For $25 \leq \alpha \leq 75$, $F_{\mathcal{T}}(\alpha) = \frac{1}{6} + \frac{\alpha - 25}{75}$ and therefore $f_{\mathcal{T}}(\alpha) = \frac{1}{75}$. Note that $F_{\mathcal{T}}(75) = \frac{5}{6}$.
- For $75 \leq \alpha \leq 100$, $F_{\mathcal{T}}(\alpha) = 1 - \frac{1}{4}(40 - 0.4\alpha)^2 \frac{1}{150}$ and therefore $f_{\mathcal{T}}(\alpha) = \frac{100 - \alpha}{1875}$.

The pdf is a trapezium increasing linearly from 0 at $\alpha = 0$ to $\frac{1}{75}$ at $\alpha = 25$; then being constant until $\alpha = 75$ and decreasing linearly to 0 at $\alpha = 100$.

4. (a) $f_{\mathcal{X}, \mathcal{Y}}(u, v) = f_{\mathcal{X}}(u)f_{\mathcal{Y}}(v) = \frac{1}{2\pi} \exp\left[-\frac{u^2 + v^2}{2}\right]$.
- (b) The region over which the joint pdf must be integrated in order to find $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\}$ is shown in the left-hand figure below.



The volume outside the circle of radius $\sqrt{2}\alpha$ can be found by changing to polar coordinates. We have $P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\} = \int_{r=\sqrt{2}\alpha}^{\infty} \int_{\theta=0}^{2\pi} \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right) r d\theta dr = \int_{r=\sqrt{2}\alpha}^{\infty} r \exp\left(-\frac{r^2}{2}\right) dr$
 $= -\exp\left(-\frac{r^2}{2}\right) \Big|_{\sqrt{2}\alpha}^{\infty} = \exp(-\alpha^2).$

- (c) $\mathcal{Z} = \mathcal{X}^2 + \mathcal{Y}^2$. From part (b), we get that $P\{\mathcal{Z} > \beta\} = 1 - F_{\mathcal{Z}}(\beta) = \exp(-\beta/2)$ for $\beta \geq 0$. Hence, $f_{\mathcal{Z}}(\beta) = \frac{1}{2} \exp(-\beta/2)$ for $\beta \geq 0$. This is an exponential density with parameter $\frac{1}{2}$.
- (d) $P\{|\mathcal{X}| > \alpha\} = 2Q(\alpha)$. Hence, $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} = P\{|\mathcal{X}| > \alpha\}P\{|\mathcal{Y}| > \alpha\} = 4Q^2(\alpha)$.
- (e) From the middle figure above, it is obvious that

$$P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} = 4Q^2(\alpha) < P\{\mathcal{X}^2 + \mathcal{Y}^2 > 2\alpha^2\} = \exp(-\alpha^2) \text{ for } \alpha > 0.$$

- (f) Taking square roots on both sides, we get $Q(x) < \frac{1}{2} \exp(-x^2/2)$ for $x > 0$.
- (g) $P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} = P\{|\mathcal{X}| < \alpha\}P\{|\mathcal{Y}| < \alpha\} = [1 - 2Q(\alpha)]^2$ is the probability that the random point lies inside the square of side 2α . As shown in the right-hand figure, this square is inscribed by the circle of radius α and circumscribed by the circle of radius $\sqrt{2}\alpha$. Hence,

$$P\{\mathcal{X}^2 + \mathcal{Y}^2 \leq \alpha^2/2\} < P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} < P\{\mathcal{X}^2 + \mathcal{Y}^2 < \alpha^2\}.$$

These inequalities are equivalent to the result: $\exp(-\alpha^2) < 4Q(\alpha) - 4Q^2(\alpha) < \exp(-\alpha^2/2)$. Now, $4Q(\alpha) - 4Q^2(\alpha) < 4Q(\alpha)$, and therefore we get $Q(x) > \frac{1}{4} \exp(-x^2)$ for $x > 0$. Note that $Q(0) = \frac{1}{2}$ while the bound equals $\frac{1}{4}$ at $x = 0$. The upper bound can also be obtained from the above result. Note that since $Q(\alpha) \leq \frac{1}{2}$ for $\alpha > 0$, it follows that $4Q(\alpha) - 4Q^2(\alpha) = 4Q(\alpha)[1 - Q(\alpha)] > 2Q(\alpha)$ for $\alpha > 0$. Hence, $\exp(-\alpha^2/2) > 4Q(\alpha) - 4Q^2(\alpha) > 2Q(\alpha)$ etc.

5. $\mathcal{Z}$ takes on values in $(0, 1)$.

$$P\{\mathcal{Z} > \alpha\} = 1 - F_{\mathcal{Z}}(\alpha) = P\{\mathcal{X}^2 \mathcal{Y} > \alpha\} = \int_{v=\alpha}^1 \int_{u=\sqrt{\alpha/v}}^1 2u du = \int_{v=\alpha}^1 1 - \alpha/v dv = v - \alpha \ln v \Big|_{\alpha}^1$$

$$= 1 - \alpha + \alpha \ln \alpha. \text{ Hence, } f_{\mathcal{Z}}(\alpha) = \begin{cases} -\ln \alpha, & 0 < \alpha < 1, \\ 0, & \text{elsewhere.} \end{cases}$$