

ECE 413: Solutions to Problem Set 12

1. (a) The marginal pmfs $p_{\mathcal{X}}(u)$ and $p_{\mathcal{Y}}(v)$ are column and row sums as shown in the table below.

$\begin{array}{c} u \rightarrow \\ v \downarrow \end{array}$	0	1	3	5	Row sum
4	0	1/12	1/6	1/12	1/3
3	1/6	1/12	0	1/12	1/3
-1	1/12	1/6	1/12	0	1/3
Column sum	1/4	1/3	1/4	1/6	1

- (b) The eyeball test tells us that $\mathcal{X}$ and $\mathcal{Y}$ are *dependent* random variables.

$$(c) P\{\mathcal{X} \leq \mathcal{Y}\} = p_{\mathcal{X},\mathcal{Y}}(0, 3) + p_{\mathcal{X},\mathcal{Y}}(0, 4) + p_{\mathcal{X},\mathcal{Y}}(1, 3) + p_{\mathcal{X},\mathcal{Y}}(1, 4) + p_{\mathcal{X},\mathcal{Y}}(3, 4) = \frac{1}{2}.$$

$$P\{\mathcal{X} + \mathcal{Y} \leq 4\} = p_{\mathcal{X}}(0) + p_{\mathcal{X},\mathcal{Y}}(1, 3) + p_{\mathcal{X},\mathcal{Y}}(1, -1) + p_{\mathcal{X},\mathcal{Y}}(3, -1) + p_{\mathcal{X},\mathcal{Y}}(5, -1) = \frac{7}{12}.$$

$$(d) p_{\mathcal{X}|\mathcal{Y}}(u|3) = \frac{p_{\mathcal{X},\mathcal{Y}}(u, 3)}{p_{\mathcal{Y}}(3)} = 1/2, 1/4, 1/4 \text{ respectively for } u = 0, 1, 5.$$

$$E[\mathcal{X}|\mathcal{Y} = 3] = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 5 \times \frac{1}{4} = \frac{3}{2}.$$

$$\text{var}(\mathcal{X}|\mathcal{Y} = 3) = E[\mathcal{X}^2|\mathcal{Y} = 3] - (E[\mathcal{X}|\mathcal{Y} = 3])^2 = 1 \times \frac{1}{4} + 5^2 \times \frac{1}{4} - \left(\frac{3}{2}\right)^2 = \frac{26}{4} - \frac{9}{4} = \frac{17}{4}.$$

2. (a) When $\mathcal{Y} = m$ is observed, it must be that $\mathcal{X} \geq m$. Hence, for $n \geq m \geq 0$,

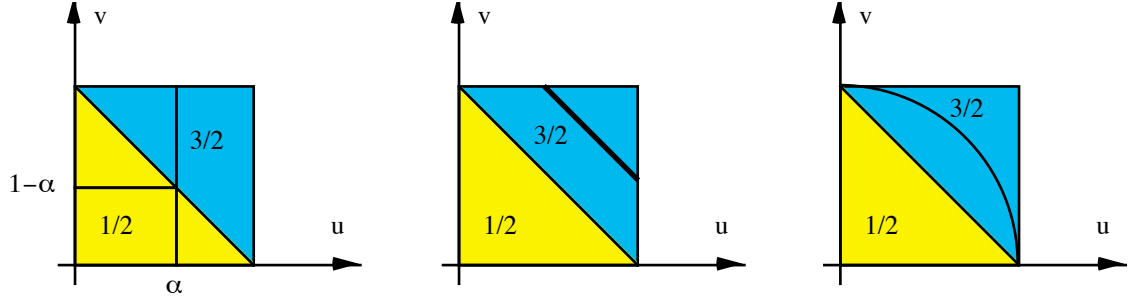
$$p_{\mathcal{X}|\mathcal{Y}}(n|m) = \frac{p_{\mathcal{Y}|\mathcal{X}}(m|n)p_{\mathcal{X}}(m)}{p_{\mathcal{Y}}(m)} = \frac{\binom{n}{m}p^m(1-p)^{n-m} \cdot (\lambda T)^n \exp(-\lambda T)/n!}{(\lambda p T)^m \cdot \exp(-\lambda p T)/m!}$$

$$= \frac{(\lambda(1-p)T)^{n-m} \exp(-\lambda(1-p)T)}{(n-m)!}.$$

This is a *displaced* Poisson pmf: the conditional pmf of $\mathcal{X}$ is that of the random variable $m + \mathcal{Z}$ where $\mathcal{Z}$ is a Poisson random variable with parameter $\lambda(1-p)T$. Note that $m + \mathcal{Z}$ takes on values $m, m+1, \dots$

- (b) $E[\mathcal{X}|\mathcal{Y} = m] = E[m + \mathcal{Z}] = m + \lambda(1-p)T.$

3. The joint pdf is as shown in the figure below.



- (a) From the left-hand figure above, $f_X(\alpha) = 0$ for $\alpha < 0$ or $\alpha > 1$, while for any $\alpha, 0 \leq \alpha \leq 1$,

$$f_X(\alpha) = \int_{-\infty}^{\infty} f_{X,Y}(\alpha, v) dv = \int_0^{1-\alpha} \frac{1}{2} dv + \int_{1-\alpha}^1 \frac{3}{2} dv = \frac{1}{2}(1-\alpha) + \frac{3}{2}\alpha = \frac{1}{2} + \alpha.$$

- (b) When the pdf has constant value over a region, we can find the probability that the random point lies in that region by finding the area of the region and multiplying by the pdf value.

$$\text{Thus, } P\{X + Y \leq 3/2\} = 1 - P\{X + Y \geq 3/2\} = 1 - \frac{3}{2} \times \left[\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \right] = \frac{13}{16}$$

$$\text{and } P\{X^2 + Y^2 \geq 1\} = \frac{3}{2} \times \left[1 - \frac{\pi}{4} \right] = \frac{3}{2} - \frac{3\pi}{8}.$$

4. The pdf is nonzero over the shaded region in the left-hand figure shown below.

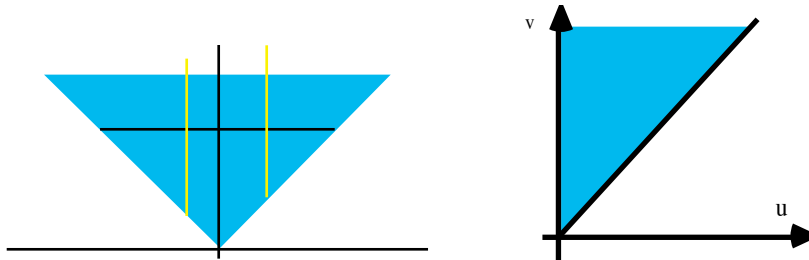
- (a) , (b) From the figure above, we get that for $v > 0$,

$$f_Y(v) = \int_{u=-v}^{u=+v} c(v^2 - u^2) \exp(-v) du = c \left[v^2 u - \frac{u^3}{3} \right] \exp(-v) \Big|_{u=-v}^{u=+v} = c \left(\frac{4}{3} \right) v^3 \exp(-v)$$

which is of the form of a *gamma* pdf with parameters (4, 1). Thus, $4c/3 = 1/\Gamma(4) = 1/3 \Rightarrow c = 1/8$.

On the other hand, for $u > 0$, $f_X(u) = \int_{v=u}^{\infty} \frac{1}{8}(v^2 - u^2) \exp(-v) dv = \frac{1}{4}(1 + u) \exp(-u)$, while if $u < 0$, the limits are $v = -u$ and ∞ . Consequently, $f_X(u) = \frac{1}{4}(1 + |u|) \exp(-u)$, $-\infty < u < \infty$.

- (c) The pdf of X is an even function of u and $\int_0^{\infty} f_X(u) du$ is finite. Hence, $E[X] = 0$.



5. (a) The joint pdf is nonzero on the shaded region shown in the right-hand figure above.

(b) For $u > 0$, $f_{\mathcal{X}}(u) = \int_v^{\infty} 2 \exp(-u-v) dv = 2 \exp(-2u)$ and $f_{\mathcal{X}}(u) = 0$ for $u < 0$.
 For $v > 0$, $f_{\mathcal{Y}}(v) = \int_0^v 2 \exp(-u-v) du = 2 \exp(-v) - 2 \exp(-2v)$ and $f_{\mathcal{Y}}(v) = 0$ for $v < 0$.

- (c) No, the eyeball test says that the random variables are dependent.

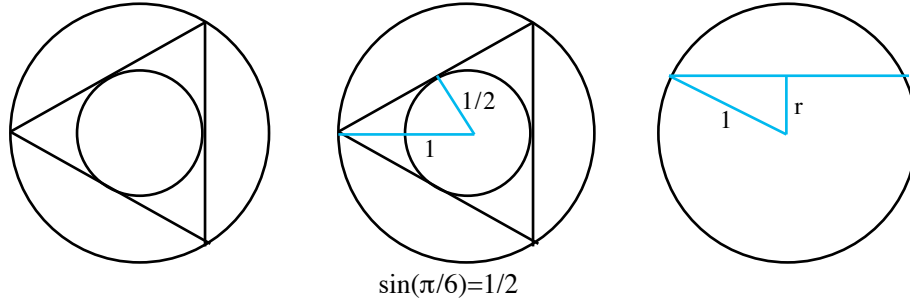
(d) $P\{\mathcal{Y} > 3\mathcal{X}\} = \int_{u=0}^{\infty} \int_{v=3u}^{\infty} 2(\exp(-u-v)) dv du = \int_{u=0}^{\infty} 2 \exp(-4u) du = \frac{1}{2}.$

- (e) For $\alpha > 0$,

$$P\{\mathcal{X} + \mathcal{Y} \leq \alpha\} = \int_{u=0}^{\alpha/2} \int_{v=u}^{\alpha-u} 2e^{-u-v} dv du = \int_{u=0}^{\alpha/2} 2e^{-u} [e^{(-u)} - e^{(-\alpha+u)}] du = (1 + \alpha) \exp(-\alpha).$$

- (f) $f_{\mathcal{Z}}(\alpha) = \frac{d}{d\alpha} F_{\mathcal{Z}}(\alpha) = \frac{d}{d\alpha} P\{\mathcal{X} + \mathcal{Y} \leq \alpha\} = \frac{d}{d\alpha} (1 + \alpha) \exp(-\alpha) = \alpha \exp(-\alpha)$ for $\alpha > 0$ and $f_{\mathcal{Z}}(\alpha) = 0$ for $\alpha < 0$. This is a *gamma* pdf with parameters $(2, 1)$.

6. The pdf of $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the unit disc, and thus has value π^{-1} .



- (a) The circle inscribed in the triangle has radius $1/2$ (cf. Ross, p. 198). If the midpoint of the chord is outside the circle, then the length $\mathcal{L}$ of the random chord is greater than the side of the equilateral triangle. *outside* the inscribed circle in the triangle. This inscribed circle has radius $1/2$ (Ross, p. 198). and hence,
 $P\{\mathcal{L} > \frac{1}{2}\} = \left[\pi \left(\frac{1}{2} \right)^2 \right] \pi^{-1} = \frac{1}{4}$

- (b) The length of a chord at distance r from the center is $2\sqrt{1-r^2}$.

Hence, $\mathcal{L} = 2\sqrt{1-\mathcal{R}^2} = 2\sqrt{1-rvX^2-, \mathcal{Y}^2}$

and $F_{\mathcal{L}}(\alpha) = P\{\mathcal{L} \leq \alpha\} = P\{\mathcal{X}^2 + \mathcal{Y}^2 \geq 1 - \alpha^2/4\} = 1 - \pi(1 - \alpha^2/4)\pi^{-1} = \alpha^2/4$.
 From this we get that $f_{\mathcal{L}}(\alpha) = \alpha/2$ for $0 \leq \alpha \leq 2$, and $f_{\mathcal{L}}(\alpha) = 0$ otherwise.

(c) $E[\mathcal{L}] = \int_0^2 \alpha \cdot \frac{\alpha}{2} d\alpha = \frac{4}{3}.$