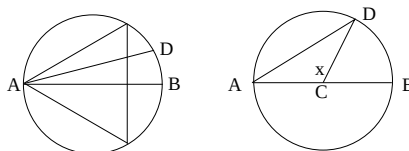


ECE 413: Solutions to Problem Set 11

1. $Y(f) = H(f)X(f) = \text{rect}(f/2) \exp(-\pi f^2)$, and $y(0) = \int_{-\infty}^{\infty} Y(f) df = \int_{-1}^1 \exp(-\pi f^2) df$. But, the integrand is the Gaussian pdf $\mathcal{N}(0, (2\pi)^{-1})$. Hence, $y(0) = \Phi(\sqrt{2\pi}) - \Phi(-\sqrt{2\pi}) = 2\Phi(\sqrt{2\pi}) - 1 \approx 0.9826$
2. (a) $\mathcal{X}$ is uniformly distributed on $[0, 2\pi)$. From the diagram below, it should be obvious that the probability that the random chord is longer than the side of the inscribed equilateral triangle is $P\{2\pi/3 < \mathcal{X} < 4\pi/3\} = \frac{1}{3}$.



- (b) Since the circle has radius 1, an arc of length $\mathcal{X}$ subtends angle $\mathcal{X}$ at the center of the circle. Furthermore, the length $\mathcal{L}$ of the chord is $2 \sin(\mathcal{X}/2)$, increasing from 0 when $\mathcal{X} = 0$ to 2 when $\mathcal{X} = \pi$ and decreasing back to 0 at $\mathcal{X} = 2\pi$. For any $x, 0 < x < 2$,

$$F_{\mathcal{L}}(x) = P\{\mathcal{L} \leq x\} = P\{2 \sin(\mathcal{X}/2) \leq x\} = 2 \cdot P\{0 \leq \mathcal{X} \leq 2 \arcsin(x/2)\} = \left(\frac{2}{\pi}\right) \arcsin\left(\frac{x}{2}\right)$$

. Hence,

$$f_{\mathcal{L}}(x) = \frac{d}{dx} F_{\mathcal{L}}(x) = \begin{cases} \frac{1}{\pi \sqrt{1-(x/2)^2}}, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

3. The area under the curve t^3 from 0 to T is $T^4/4$. Thus, $F(t) = 1 - \exp(-t^4/4)$.
 - (a) $P\{\mathcal{X} > 2\} = \exp(-16/4) = \exp(-4)$.
 - (b) $P\{0.4 < \mathcal{X} < 1.4\} = \exp(-0.0256/4) - \exp(-(1.4)^4/4)$.
 - (c) $P\{\mathcal{X} > 2 | \mathcal{X} > 1\} = P\{\mathcal{X} > 2\} / P\{\mathcal{X} > 1\} = \exp(-4) / \exp(-1/4) = \exp(-3.75)$.
4. As discussed in class, the first arrival time $\mathcal{X}$ is an exponential random variable with parameter λ , and its pdf is $\lambda \exp(-\lambda t)$ for $t > 0$.

- (a) Hence, $P\{\mathcal{X} \leq T\} = \int_0^T \lambda \exp(-\lambda t) dt = 1 - \exp(-\lambda T) = \frac{\lambda T}{1!} - \frac{(\lambda T)^2}{2!} + \dots$

- (b) $P(A) = P\{N(0, T] = 1\} = \frac{\lambda T}{1!} \exp(-\lambda T) = \lambda T \exp(-\lambda T)$.

- (c) $P(A)$ in part (b) is *not* the same as $P\{\mathcal{X} \leq T\}$ from part (a). In fact, $A \subset \{\mathcal{X} \leq T\}$ since the latter event includes the possibility that the *second* arrival also occurred during $(0, T]$.

- (d) For $0 < t < T$,

$$\begin{aligned} P(\{X \leq t\} | A) &= \frac{P(\{X \leq t\} \cap A)}{P(A)} = \frac{P(\{N(0, t] = 1\} \cap \{N(t, T] = 0\})}{P\{N(0, T] = 1\}} \\ &= \frac{P\{N(0, t] = 1\} \times P\{N(t, T] = 0\}}{P\{N(0, T] = 1\}} \text{ by independence of disjoint intervals} \\ &= \frac{(\lambda t) \exp(-\lambda t) \times \exp(-\lambda(T-t))}{\lambda T \exp(-\lambda T)} = \frac{t}{T}. \end{aligned}$$

Thus, the *conditional* pdf of the first arrival time $\mathcal{X}$ given that there was exactly one arrival in $(0, T]$ is a *uniform* pdf on $(0, T]$!!

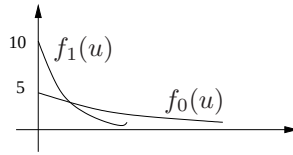
5. I'm considering, I'm considering!

- (a) The number of arrivals in $(0, 4]$ is $N(0, 4]$, a Poisson random variable with parameter 4λ . Hence, $E[N(0, 4)] = 4\lambda$.
- (b) The event $\{N(0, 3] = 3\} \cap \{N(2, 6] = 0\}$ is exactly the same as $\{N(0, 2] = 3\} \cap \{N(2, 6] = 0\}$ and thus $P[\{N(0, 3] = 3\} \cap \{N(2, 6] = 0\}] = P[\{N(0, 2] = 3\} \cap \{N(2, 6] = 0\}]$. We get

$$P[\{N(0, 3] = 3\} \cap \{N(2, 6] = 0\}] = \frac{(2\lambda)^3}{3!} \exp(-2\lambda) \times \exp(-4\lambda) = \frac{4\lambda^3}{3} \exp(-6\lambda)$$

by independence of Poisson variables on disjoint intervals of time.

- (c) The number of arrivals in $(0, 6]$ is $N(0, 6]$, a Poisson random variable with parameter 6λ . If it is observed that $N(0, 6] = k$, then the maximum-likelihood estimate of the Poisson parameter 6λ is k (cf. Problem 3(c) of Problem Set 4), and hence the maximum-likelihood estimate of the arrival rate is $k/6 = \#$ of arrivals/length of interval which is exactly what one would expect.
- (d) $P\{N(0, t) \geq 1\} = 1 - P\{N(0, t) = 0\} = 1 - \exp(-\lambda t) = 1 - 2^{-t}$ when $\lambda = \ln 2$.
6. (a) The pdfs are as shown below.



(b) $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{10 \cdot \exp(-10u)}{5 \cdot \exp(-5u)} = 2 \cdot \exp(-5u)$

which has value 2 at $u = 0$ and decays away to 0 as $u \rightarrow \infty$. Note that $\Lambda(u) > 1$ for $u < 0.2 \ln 2$. Thus, the likelihood ratio test is equivalent to deciding in favor of H_1 if the observed value of $\mathcal{X}$ is *smaller* than the threshold $0.2 \ln 2$. Equivalently, $\Gamma_1 = (0, 0.2 \ln 2)$, $\Gamma_0 = (0.2 \ln 2, \infty)$.

(c) $P_{FA} = \int_{\Gamma_1} f_0(u) du = \int_0^{0.2 \ln 2} 5 \cdot \exp(-5u) du = -\exp(-5u) \Big|_0^{0.2 \ln 2} = -\frac{1}{2} - (-1) = \frac{1}{2}$.

$P_{MD} = \int_{\Gamma_0} f_1(u) du = \int_{0.2 \ln 2}^{\infty} 10 \cdot \exp(-10u) du = -\exp(-10u) \Big|_{0.2 \ln 2}^{\infty} = 0 - (-\exp(-2 \ln 2)) = \frac{1}{4}$.

- (d) $\Lambda(u) = 2 \cdot \exp(-5u) > \frac{\pi_0}{\pi_1}$ for $u < 0.2 \ln \left(\frac{2\pi_1}{\pi_0} \right) = 0.2 \ln 2 + 0.2 \ln \left(\frac{\pi_1}{\pi_0} \right) = \xi$. Thus, the minimum-error-probability decision rule is equivalent to deciding in favor of H_1 if the observed value of $\mathcal{X}$ is smaller than ξ . Note that $\xi < 0$ if $\pi_0 > 2\pi_1$, that is, if $\pi_0 > 2/3$.

- (e) If $\pi_0 = 1/3$, then $\xi = 0.2 \ln 4$. Hence,

$$P_{FA} = \int_0^{0.2 \ln 4i} 5 \cdot \exp(-5u) du = -\exp(-5u) \Big|_0^{0.2 \ln 4i} = -\frac{1}{4} - (-1) = \frac{3}{4}$$

$$P_{MD} = \int_{0.2 \ln 4i}^{\infty} 10 \cdot \exp(-10u) du = -\exp(-10u) \Big|_{\xi}^{\infty} = 0 - (-\exp(-2 \ln 4i)) = \frac{1}{16}$$

The average error probability thus is $\bar{P}_e = \frac{1}{3} P_{FA} + \frac{2}{3} P_{MD} = \frac{7}{24}$. Note that since $\pi_0 < \pi_1$, the Bayesian decision rule allows P_{FA} to increase in return for a decrease in P_{MD} because the latter is weighted more heavily.

- (f) If the decision rule always decides H_1 is the true hypothesis it makes errors if and only if H_0 is the true hypothesis. Hence, $\bar{P}_e = \pi_0$.
- (g) When $\pi_0 > 2/3$, the threshold ξ is less than 0. Since $\mathcal{X}$ takes on nonnegative values, it is always larger than the threshold, and hence the decision is always H_0 . The average error probability is π_1 , and since this is the minimum-error-probability rule, we cannot do any better than this. Note that $\pi_1 < 1/3$.

When $\pi_0 > 2/3$, it follows that $\pi_0 > 2\pi_1$. The average probability of error for the maximum-likelihood rule is $\pi_0 \cdot (1/2) + \pi_1 \cdot (1/4) > 2\pi_1 \cdot (1/2) + \pi_1 \cdot (1/3) = 1.25\pi_1$.