

ECE 413: Solutions to Problem Set 9

1. (a) He sells all H newspapers if the demand $\mathcal{X}$ is *at least* H , that is, $\mathcal{X} \geq H$. Hence,

$$P\{\text{sells all papers}\} = P\{\mathcal{X} \geq H\} = \sum_{k=H}^{\infty} p_{\mathcal{X}}(k) = 1 - F_{\mathcal{X}}(H^-) = 1 - F_{\mathcal{X}}(H-1).$$

$$(b) \mathcal{Z} = \begin{cases} (c_3 - c_2)\mathcal{X} + (c_1 - c_2)(H - \mathcal{X}) = (c_3 - c_1)\mathcal{X} + (c_1 - c_2)H & \text{if } \mathcal{X} \leq H, \\ (c_3 - c_2)H & \text{if } \mathcal{X} \geq H. \end{cases}$$

Note that the first formula reduces to the second if $\mathcal{X} = H$.

$$(c) E[\mathcal{Z}] = \sum_{k=0}^H [(c_3 - c_1)\mathcal{X} + (c_1 - c_2)H] p_{\mathcal{X}}(k) + \sum_{k=H+1}^{\infty} [(c_3 - c_2)H] p_{\mathcal{X}}(k) = g(H).$$

- (d) The extra paper (over and above the usual H) gives an *additional* profit of $(c_3 - c_2)$ if $\mathcal{X} > H$ (which occurs with probability $1 - F_{\mathcal{X}}(H)$), and a loss of $(c_2 - c_1)$ if $\mathcal{X} \leq H$ (which occurs with probability $F_{\mathcal{X}}(H)$). Hence, he makes an average *additional* profit of

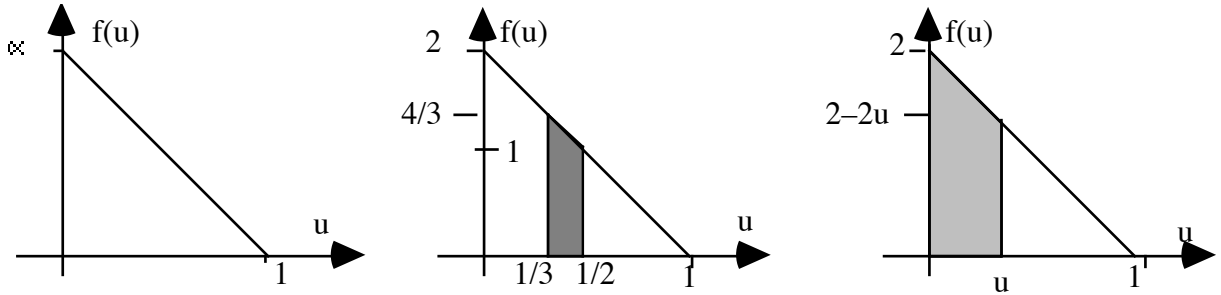
$$A(H) = (c_3 - c_2)(1 - F_{\mathcal{X}}(H)) - (c_2 - c_1)F_{\mathcal{X}}(H) = (c_3 - c_2) - (c_3 - c_1)F_{\mathcal{X}}(H).$$

$$(e) \dots \leq F_{\mathcal{X}}(H-1) \leq F_{\mathcal{X}}(H) \leq F_{\mathcal{X}}(H+1) \leq \dots \Rightarrow \dots \geq A(H-1) \geq A(H) \geq A(H+1) \geq \dots$$

- (f) As H increases and $F_{\mathcal{X}}(H) \rightarrow 1$, $A(H) \rightarrow c_1 - c_2 < 0$. Hence, $A(H) < 0$ for large values of H .

- (g) $A(H) \leq 0$ if and only if $F_{\mathcal{X}}(H) \geq (c_3 - c_2)/(c_3 - c_1)$. Let H_0 denote the smallest integer such that $A(H_0) \leq 0$. Then, buying the $(H_0 + 1)$ -th paper will *reduce* the total profit, and thus H_0 is the number of papers that will maximize the average daily profit. Note that if $A(H_0) = 0$ exactly, the average additional profit from the $(H_0 + 1)$ -th paper is 0: why work harder than you have to?

2. The probability density function is as shown in the leftmost figure below.



- (a) Since the area under the curve is obviously $\alpha/2$ and should equal 1, we conclude that $\alpha = 2$.
 $P\{6\mathcal{X}^2 > 5\mathcal{X} - 1\} = P\{6\mathcal{X}^2 - 5\mathcal{X} + 1 > 0\} = P\{(3\mathcal{X} - 1)(2\mathcal{X} - 1) > 0\}$
 $= P\left\{\left\{\mathcal{X} > \frac{1}{3}\right\} \cap \left\{\mathcal{X} > \frac{1}{2}\right\}\right\} + P\left\{\left\{\mathcal{X} < \frac{1}{3}\right\} \cap \left\{\mathcal{X} < \frac{1}{2}\right\}\right\} = P\left\{\left\{\mathcal{X} > \frac{1}{2}\right\} \cup \left\{\mathcal{X} < \frac{1}{3}\right\}\right\}$
 $= 1 - P\left\{\frac{1}{3} \leq \mathcal{X} \leq \frac{1}{2}\right\}$. Now, $P\left\{\frac{1}{3} \leq \mathcal{X} \leq \frac{1}{2}\right\}$ is the area under the pdf between $\frac{1}{3}$ and $\frac{1}{2}$ as shown in the middle figure above. This can be calculated as the *difference* between the areas of two triangles of base $\frac{2}{3}$ and $\frac{1}{2}$ respectively and altitudes $\frac{4}{3}$ and 1 respectively. Hence, we get $P\{6\mathcal{X}^2 > 5\mathcal{X} - 1\} = 1 - P\left\{\frac{1}{3} \leq \mathcal{X} \leq \frac{1}{2}\right\} = \frac{29}{36}$.
- (b) $F_{\mathcal{X}}(u)$ is the area under the pdf curve to the left of the point u . Clearly, $F_{\mathcal{X}}(u) = 0$ for $u < 0$. For $0 \leq u \leq 1$, the area is as shown in the rightmost figure above (difference of two triangles again!), and we get $F_{\mathcal{X}}(u) = 2u - u^2$. For $u \geq 1$, $F_{\mathcal{X}}(u) = 1$ obviously. Putting all these together, we have

$$F_{\mathcal{X}}(u) = \begin{cases} 0, & u < 0, \\ 2u - u^2, & 0 \leq u \leq 1, \\ 1, & u > 1. \end{cases}$$

3. (a) No, if $\mathcal{X} = 0.68$, the gas station *cannot* satisfy the demand that week. On the other hand, if $\mathcal{X} = 0.43$ then the gas station *can* satisfy the demand that week.

(b) $P\{\text{demand can be satisfied}\} = P\left\{\mathcal{X} \leq \frac{1}{2}\right\} = \int_0^{0.5} 5(1-u)^4 du = -(1-u)^5 \Big|_0^{0.5} = \frac{31}{32}.$

(c) We want the minimum value of C such that $P\{\mathcal{X} > C\} \leq 10^{-5}$. But,

$$P\{\mathcal{X} > C\} = \int_C^1 5(1-u)^4 du = -(1-u)^5 \Big|_C^1 = (1-C)^5 \leq 10^{-5} \text{ if } C \geq 0.9.$$

(d) $\mathcal{Y} = \min\{\mathcal{X}, C\} = \begin{cases} \mathcal{X}, & \text{if } \mathcal{X} \leq C, \\ C, & \text{if } \mathcal{X} > C. \end{cases}$

(e) The weekly *gross* profit is $\$640\mathcal{Y}$ and the *average* weekly gross profit is

$$\mathbb{E}[640\mathcal{Y}] = \int_0^C 5(1-u)^4(640u) du + \int_C^1 5(1-u)^4(640C) du = \frac{640}{6} [1 - (1-C)^6].$$

As a function of C , this increases from 0 at $C = 0$ to $640/6$ when $C = 1$. The average *net* profit is $\mathbb{E}[(640\mathcal{Y} - 20C)] = \mathbb{E}[640\mathcal{Y}] - 20C = \frac{640}{6} [1 - (1-C)^6] - 20C$ whose maximum can be found by the usual calculus method to be at $C = \frac{1}{2}$. Thus, a 500 gallon tank that costs \$10 per week to rent gives an average gross profit of $\$630/6$ and the maximum average net profit of $\$570/6$ per week. In contrast, a full-sized 1000 gallon tank gives only a slightly larger gross profit of $\$640/6$ per week, but a smaller net profit of $\$520/6$ per week. Of course, there may be intangible losses in goodwill if the tank is emptied before the week ends. However, the probability of this happening is only $\frac{1}{32}$ when $C = \frac{1}{2}$.

4. (a) $P\{\mathcal{X} \leq \ln 2\} = \int_{-\infty}^{\ln 2} f_{\mathcal{X}}(u) du = \frac{1}{2} \left[1 + \int_0^{\ln 2} \exp(-u) du \right] = \frac{1}{2} \left[1 + \left(1 - \frac{1}{2} \right) \right] = \frac{3}{4}.$

(b) $P\{|\mathcal{X}| \leq \ln 2\} = \frac{P\{|\mathcal{X}| \leq \ln 2\} \cap \{\mathcal{X} \leq \ln 2\}}{P\{\mathcal{X} \leq \ln 2\}} = \frac{P\{|\mathcal{X}| \leq \ln 2\}}{3/4} = \frac{1/2}{3/4} = \frac{2}{3}.$

(c) $\cos(\pi\mathcal{X}/2) < 0$ if $\mathcal{X}$ lies in one of the intervals $\dots, (-7, -5), (-3, -1), (1, 3), (5, 7), \dots$. Hence,

$$\begin{aligned} P\{\cos(\pi\mathcal{X}/2) < 0\} &= 2 \sum_{k=0}^{\infty} P\{4k+1 < \mathcal{X} < 4k+3\} = \sum_{k=0}^{\infty} (\exp(-4k-1) - \exp(-4k-3)) \\ &= [\exp(-1) - \exp(-3)] \sum_{k=0}^{\infty} \exp(-4k) = \frac{\exp(-1) - \exp(-3)}{1 - \exp(-4)} \\ &= \frac{\sinh(1)}{\sinh(2)} = \frac{1}{\exp(1) + \exp(-1)} = \frac{1}{2 \cosh(1)} = 0.324\dots \end{aligned}$$

5. (a) $\mathbb{E}[\mathcal{Y}] = \mathbb{E}[\mathcal{X}^2] = \int_{-1}^{+1} u^2 \frac{1}{2} du = \frac{1}{3}, \quad \mathbb{E}[\mathcal{Y}^2] = \mathbb{E}[\mathcal{X}^4] = \int_{-1}^{+1} u^4 \frac{1}{2} du = \frac{1}{5}.$

Hence, $\text{var}(\mathcal{Y}) = \mathbb{E}[\mathcal{Y}^2] - (\mathbb{E}[\mathcal{Y}])^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$

(b) $\mathbb{E}[\mathcal{Z}] = \mathbb{E}[g(\mathcal{X})] = \int_{-1}^0 -u^2 \frac{1}{2} + \int_0^{+1} u^2 \frac{1}{2} du = -\frac{1}{6} + \frac{1}{6} = 0.$

(c) $\mathcal{U}$ takes on integer values ≥ 1 . $\mathcal{V} = \sin(\pi\mathcal{U}/2) = 0$ if $\mathcal{U}$ is even. On the other hand, if $\mathcal{U} = 4k+1$, then $\mathcal{V} = +1$ while if $\mathcal{U} = 4k-1$, then $\mathcal{V} = -1$. We get

$$\mathbb{E}[\mathcal{V}] = \sum_{k=1}^{\infty} \sin(\pi k/2) \left(\frac{1}{2}\right)^k = \frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \dots = \frac{1}{2} [1 - x + x^2 - \dots] \text{ where } x = \frac{1}{4} \Rightarrow \mathbb{E}[\mathcal{V}] = \frac{2}{5}.$$

6. (a) $\mathbb{E}[\mathcal{R}] = \int_0^1 u \cdot 2u du = \frac{2}{3}, \quad \mathbb{E}[\mathcal{V}] = \int_0^1 \frac{4\pi}{3} u^3 \cdot 2u du = \frac{8\pi}{15}, \quad \mathbb{E}[\mathcal{A}] = \int_0^1 4\pi u^2 \cdot 2u du = 2\pi.$ The *average sphere* does not have average volume or average surface area. More generally, $\mathbb{E}[g(\mathcal{X})] \neq g(\mathbb{E}[\mathcal{X}]).$

(b) For $0 < u < 1$, $u^n > u^{n+1}$. Hence, $\int_0^1 u^n \cdot f_{\mathcal{R}}(u) du = \mathbb{E}[\mathcal{R}^n] > \int_0^1 u^{n+1} \cdot f(u) du = \mathbb{E}[\mathcal{R}^{n+1}].$