

## ECE 413: Solutions to Problem Set 8

1. (a)  $P(\text{wins on first roll}) = \frac{6}{36} + \frac{2}{36} = \frac{2}{9}$ .  $P(\text{loses on first roll}) = \frac{1}{36} + \frac{2}{36} + \frac{1}{36} = \frac{1}{9}$ .  
 $P(\text{establishes point } i \text{ on first roll}) = \frac{6 - |7 - i|}{36}, i = 4, 5, 6, 8, 9, 10$ . Check: probabilities sum to 1.

- (b) If the shooter's point is  $i$ , then  $P(\text{makes point } i \mid \text{point is } i) = \frac{6 - |7 - i|}{36}$  while  $P(\text{craps out}) = \frac{6}{36}$  so that  $P(\text{game ends}) = \frac{12 - |7 - i|}{36}$ . Else, the shooter rolls the dice again. Now, given disjoint events  $A$  and  $B$ , the probability that  $A$  occurs before  $B$  is the probability that a compound event of the form  $(C, C, \dots, C, A)$  occurs where  $C = (A \cup B)^c$ . Thus,

$$P(A \text{ occurs before } B) = P(A) + P(C)P(A) + [P(C)]^2P(A) + \dots = \frac{P(A)}{1 - P(C)} = \frac{P(A)}{P(A) + P(B)}.$$

Hence,  $P(\text{shooter wins} \mid \text{point is } i) = \frac{6 - |7 - i|}{12 - |7 - i|}$  which equals  $\frac{3}{9}, \frac{4}{10}, \frac{5}{11}, \frac{5}{11}, \frac{4}{10}, \frac{3}{9}$  according as  $4, 5, 6, 8, 9, 10$ . The theorem of total probability now gives that

$$P(\text{shooter wins}) = \frac{2}{9} + 2 \left[ \frac{3}{9} \times \frac{3}{36} + \frac{4}{10} \times \frac{4}{36} + \frac{5}{11} \times \frac{5}{36} \right] = \frac{244}{495} = 0.492929 \dots$$

- (c) Conditioned on the shooter's point being  $i$ , the number of *additional* dice rolls till the game ends is a geometric random variable with parameter  $p = P(\text{game ends}) = \frac{12 - |7 - i|}{36}$  and conditional mean  $\frac{36}{12 - |7 - i|}$ . The expected number of dice rolls in a game of craps is thus

$$E[\mathcal{X}] = \frac{3}{9} \times 1 + 2 \left[ \left(1 + \frac{36}{9}\right) \frac{3}{36} + \left(1 + \frac{36}{10}\right) \frac{4}{36} + \left(1 + \frac{36}{11}\right) \frac{5}{36} \right] = 3 \frac{62}{165}.$$

- (d) If the shooter's point is 8, then his chances of making the point the hard way are  $\frac{1}{36}$  while the probability of making the point the easier way (or crapping out) are  $\frac{6+4}{36}$ . Thus, the shooter's probability of winning the hard way is  $\frac{1}{1+10} = \frac{1}{11}$ . Thus, the odds are perfectly fair. Ten times out of eleven, you will lose the dollar that you wager while one time out of eleven, you will win ten dollars.

2. (a)  $F(u) = \begin{cases} 0 & u < 0, \\ u^2, & 0 \leq u < 1, \\ 1, & u \geq 1. \end{cases}$  is a valid CDF.  $P\{|\mathcal{X}| > 0.5\} = P\{\mathcal{X} > 0.5\} = 1 - F(0.5) = \frac{3}{4}$ .

- (b)  $F(u) = \begin{cases} 0 & u < 1, \\ 2u - u^2, & 1 \leq u \leq 2, \\ 1, & u > 2. \end{cases}$  is *not* a valid CDF since  $F(1) = 1 > F(2) = 0$ .

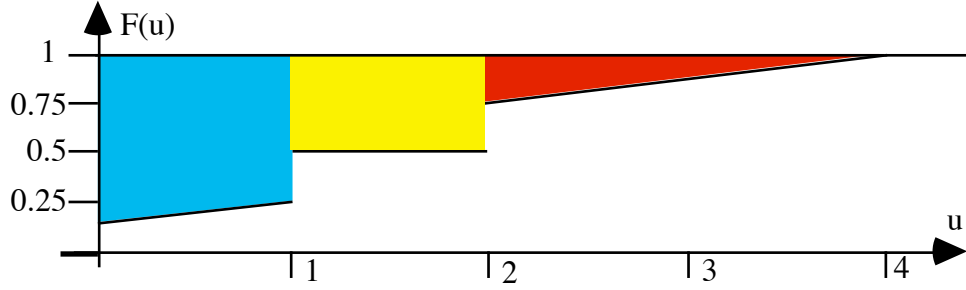
- (c)  $F(u) = \begin{cases} \frac{1}{2} \exp(2u) & u \leq 0, \\ 1 - \frac{1}{4} \exp(-3u), & u > 0, \end{cases}$  is *not* a valid CDF since it is not right-continuous at 0.

- (d)  $F(u) = \begin{cases} \frac{1}{2} \exp(2u) & u < 0, \\ 1 - \frac{1}{4} \exp(-3u), & u \geq 0, \end{cases}$  is a valid CDF.

$$P\{|\mathcal{X}| > 0.5\} = 1 - P\{|\mathcal{X}| \leq 0.5\} = 1 - (F(0.5) - F(-0.5)) = \frac{1}{2} \exp(-1) - \frac{1}{4} \exp(-1.5).$$

3. (a)  $P\{\mathcal{X} = 2\} = F_{\mathcal{X}}(2^+) - F_{\mathcal{X}}(2^-) = \frac{1}{4}$ ,  $P\{\mathcal{X} < 2\} = F_{\mathcal{X}}(2^-) = \frac{1}{2}$ ,  $P\{\mathcal{X} > 2\} = 1 - F_{\mathcal{X}}(2) = \frac{1}{4}$ ,  
 $P\{1 \leq \mathcal{X} \leq 3\} = F_{\mathcal{X}}(3) - F_{\mathcal{X}}(1^-) = \frac{7}{8} - \frac{1}{4} = \frac{5}{8}$ , and  $P\{\mathcal{X} > 2 \mid \mathcal{X} > 0\} = \frac{P\{\mathcal{X} > 2\}}{P\{\mathcal{X} > 0\}} = \frac{2}{7}$ .

- (b)  $E[\mathcal{X}]$  is the area between the CDF and the line at height 1 as shown in the figure below. Elementary geometry gives  $E[\mathcal{X}] = \frac{1}{2} \times 1 \times \left(\frac{7}{8} + \frac{6}{8}\right) + \frac{1}{2} + \frac{1}{2} \times 2 \times \frac{1}{4} = \frac{22}{16} = 1.375$ .



4. (a) We break the integral into a sum of integrals over the intervals  $[k, k+1)$  to get

$$E[\mathcal{X}] = \int_0^\infty P\{\mathcal{X} > u\} du = \sum_{k=0}^\infty \int_k^{k+1} P\{\mathcal{X} > u\} du.$$

But, for each  $u \in [k, k+1)$ ,  $P\{\mathcal{X} > u\}$  has the same value  $P\{\mathcal{X} > k\}$ . The  $k$ -th integral thus has a constant integrand  $P\{\mathcal{X} > k\}$  and therefore evaluates to  $P\{\mathcal{X} > k\}$ . Furthermore, if  $i = k+1$ , then  $P\{\mathcal{X} > k\} = P\{X \geq k+1\} = P\{X \geq i\}$ . Hence  $E[\mathcal{X}] = \sum_{k=0}^\infty P\{X > k\} = \sum_{i=1}^\infty P\{X \geq i\}$ .

- (b) For  $k = 0, 1, 2, \dots$ ,  $P\{\mathcal{X} > k\} = q^k$  where  $q = 1 - p$ . Therefore,

$$E[\mathcal{X}] = \sum_{k=0}^\infty P\{X > k\} = \sum_{k=0}^\infty q^k = \frac{1}{1-q} = \frac{1}{p}.$$

- (c)  $\sum_{k=0}^\infty k \cdot P\{\mathcal{X} > k\} = \sum_{k=0}^\infty k \sum_{l=k+1}^\infty p_{\mathcal{X}}(l)$ . Now, for any given  $l$ , the term  $p_{\mathcal{X}}(l)$  will appear only in the sums for  $k = 0, 1, \dots, l-1$ . Hence, (using LOTUS backwards!) we can write the sum as  $\sum_{k=0}^\infty k \cdot P\{\mathcal{X} > k\} = \sum_{l=0}^\infty p_{\mathcal{X}}(l) \sum_{k=0}^{l-1} k = \sum_{l=0}^\infty p_{\mathcal{X}}(l) \cdot \frac{l(l-1)}{2} = \frac{1}{2} E[\mathcal{X}(\mathcal{X} - 1)]$ .

5. If  $f(u)$  is a nonnegative (or nonpositive) function with finite area  $A$ , then  $A^{-1} \cdot f(u)$  is a valid pdf. This does not work if  $f(u)$  takes on both positive and negative values.

- (a)  $f(u) = 2u$ ,  $0 < u < 1$  is a valid pdf.  
 (b)  $f(u) = |u|$ ,  $|u| < \frac{1}{2}$  is not a valid pdf, but  $4 \cdot f(u)$  is.  
 (c)  $f(u) = 1 - |u|$ ,  $|u| < 1$  is a valid pdf.  
 (d)  $f(u) = \ln u$ ,  $0 < u < 1$  is not a valid pdf but  $-f(u)$  is.  
 (e)  $f(u) = \ln u$ ,  $0 < u < 2$  is not a valid pdf, nor is  $C \cdot f(u)$  a valid pdf for any choice of  $C$ .  
 (f)  $f(u) = \frac{2}{3}(u-1)$ ,  $0 < u < 3$ , is not a valid pdf, nor is  $C \cdot f(u)$  a valid pdf for any choice of  $C$ .  
 (g)  $f(u) = \exp(-2u)$ ,  $u > 0$  is not a valid pdf but  $2 \cdot f(u)$  is.  
 (h)  $f(u) = 4e^{-2u} - e^{-u}$ ,  $u > 0$ , is not a valid pdf, nor is  $C \cdot f(u)$  a valid pdf for any choice of  $C$ .  
 (i)  $f(u) = \exp(-|u|)$ ,  $|u| < 1$  is not a valid pdf but  $e/[2(e-1)] \cdot f(u)$  is.

6. (a)  $P\{\mathcal{X} > 20\} = \int_{20}^\infty \frac{10}{u^2} du = \left. \frac{-10}{u} \right|_{20}^\infty = \frac{1}{2}$ .

- (b)  $F_{\mathcal{X}}(u_0) = \int_{-\infty}^{u_0} f_{\mathcal{X}}(u) du = \int_{10}^{u_0} \frac{10}{u^2} du = \left. \frac{-10}{u} \right|_{10}^{u_0} = 1 - \frac{10}{u_0}$ . Note that  $F_{\mathcal{X}}(20) = \frac{1}{2}$ ; it should be!

- (c)  $P\{\mathcal{X} > 15\} = 1 - F_{\mathcal{X}}(15) = \frac{10}{15} = \frac{2}{3}$ . Assuming that the device failures are independent, the probability that at least 3 of the 6 last for 15 hours (i.e., at most 3 fail) is a binomial probability:

$$P\{\text{at least 3 of 6 devices work for 15 hours}\} = P\{\text{at most 3 failures}\} = \sum_{i=0}^3 \binom{6}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{6-i}.$$