

ECE 413: Solutions to Problem Set 7

1. (a) $P(4-0) = p^4$; $P(0-4) = q^4$. $P(4-1) = \binom{4}{1}p^4q$; $P(1-4) = \binom{4}{1}pq^4$ since the loser's single point must be one of the first 4 points, not the last (winning) point. Similarly $P(4-2) = \binom{5}{2}p^4q^2$; $P(2-4) = \binom{5}{2}p^2q^4$. As a sanity check, add the seven probabilities: $p^4 + 4p^4q + 10p^4q^2 + 20p^3q^3 + 10p^2q^4 + 4pq^4 + q^4$ and verify that the sum is 1. Mathematica is helpful here.

(b) $P(A \text{ wins}) = p^4 + 4p^4q + 10p^4q^2 + \frac{p^2}{1-2pq}(20p^3q^3) = p^4 + 4p^4q + \frac{10p^4q^2}{1-2pq} = f(p)$.

- (c) If A wins points with probability $p = 0$, he will not win any games either! So, $f(0) = 0$. Similarly, if A wins points with probability $p = 1$, he wins all games too. Hence, $f(1) = 1$. Finally, $f(1/2) = 1/2$ since the players are perfectly matched. The function $f(p)$ obtained in part (b) satisfies all these properties, but it is not a linear function of p .

(d) $\left. \frac{d}{dp}f(p) \right|_{p=\frac{1}{2}} = 4p^3 + 16p^3q - 4p^4 + \frac{(1-2pq)(40p^3q^2 - 20p^4q) - (-2q + 2p)10p^4q^2}{(1-2pq)^2} \Big|_{p=\frac{1}{2}} = 2.5$ and

so $f(\frac{1}{2} + \epsilon) = \frac{1}{2} + 2.5\epsilon + \dots$ so that small differences between the *point win* probabilities are amplified in the *game win* probabilities. If A has a 51% chance of winning each point, he has a 52.5% chance of winning a game. Such amplification is much stronger for *set win* probabilities and *match win* probabilities. See "The Drunken Tennis Player" in Ian Stewart's *Game, Set, and Math*, Penguin Books, 1981.

(e) $E[\mathcal{X}] = 4(p^4 + q^4) + 5(4p^4q + 4pq^4) + 6(10p^4q^2 + 10p^2q^4) + \left(6 + \frac{2}{p^2 + q^2}\right)20p^3q^3 = 6\frac{3}{4}$ at $p = \frac{1}{2}$.

Exercise: show that as a function of p , $E[\mathcal{X}]$ is maximum at $p = \frac{1}{2}$.

2. (a) The bill passes if the Conservative Republicans and at least one other group votes for it, i.e. the event $A \cap (B \cup C \cup D)$ occurs, or if all groups except the Conservative Republicans vote for it, i.e. the event $A^c \cap B \cap C \cap D$ occurs. Note that these are disjoint events (Why?). Now, $(B \cup C \cup D)^c = B^c \cap C^c \cap D^c$ by DeMorgan's theorem, and thus we can partition A into two disjoint events as follows: $A = A \cap (B \cup C \cup D) \cup (A \cap (B^c \cap C^c \cap D^c))$. Hence, we see that

$$P(A \cap (B \cup C \cup D)) = P(A) - P(A \cap B^c \cap C^c \cap D^c) = P(A) - P(A)P(B^c)P(C^c)P(D^c)$$

by independence. Since $P(A^c \cap B \cap C \cap D) = P(A^c)P(B)P(C)P(D)$ by independence, we get

$$\begin{aligned} P(\text{bill passes}) &= P(A \cap (B \cup C \cup D)) + P(A^c \cap B \cap C \cap D) \\ &= P(A) - P(A)P(B^c)P(C^c)P(D^c) + P(A^c)P(B)P(C)P(D) \\ &= 0.9 \times [1 - 0.4 \times 0.5 \times 0.8] + 0.1 \times 0.6 \times 0.5 \times 0.2 = 0.756 + 0.006 = 0.762. \end{aligned}$$

- (b) The Conservative Republicans must be in favor for the motion to override to pass. Furthermore, since politics makes strange bedfellows, either the Liberal Democrats must also be in favor or both the other groups must also be in favor. Hence,

$$\begin{aligned} P(\text{veto overridden}) &= P(E \cap (H \cup (F \cap G))) \\ &= P(E) \times P(H \cup (F \cap G)) \quad \text{by independence of } E \text{ and } H \cup (F \cap G) \\ &= P(E) \times [P(H) + P(F \cap G) - P(F \cap G \cap H)] \\ &= P(E) \times [P(H) + P(F)P(G) - P(F)P(G)P(H)] \quad \text{independence again!} \\ &= 0.99 \times [0.1 + 0.4 \times 0.6 - 0.6 \times 0.4 \times 0.1] = 0.31284. \end{aligned}$$

3. The likelihood matrix and joint probability matrices are as shown below, and the ML and MAP decision rules are indicated by boldface entries.

Hypothesis	$\mathcal{Y} = 1$	$\mathcal{Y} = 2$	$\mathcal{Y} = 3$	$P(H_i)$	$\mathcal{Y} = 1$	$\mathcal{Y} = 2$	$\mathcal{Y} = 3$
$H_1 : \mathcal{X} = 1$	0.8	0.1	0.1	0.5	0.4	0.05	0.05
$H_2 : \mathcal{X} = 2$	0.05	0.9	0.05	0.25	0.0125	0.225	0.0125
$H_2 : \mathcal{X} = 3$	0.15	0.05	0.8	0.25	0.0375	0.0125	0.2

Thus, the ML and MAP rules are the same in this case. Other *a priori* probabilities could make the MAP rule different from the ML rule ...

4. (a) The likelihood of a hit is $P(H|F) = \frac{2}{5}$ or $P(H|C) = \frac{1}{4}$, or $P(H|S) = \frac{1}{6}$ depending on which hypothesis is true. Since $P(H|F) = \frac{2}{5}$ is the largest, the maximum-likelihood decision is that it was a fastball.
- (b) We know from Problem Set 6 that $P(F) = \frac{5}{24}$, $P(C) = \frac{10}{24}$ and $P(S) = \frac{9}{24}$. Now we compare the joint probabilities

$$P(H|F)P(F) = \frac{2}{5} \times \frac{5}{24} = \frac{2}{24}, P(H|C)P(C) = \frac{1}{4} \times \frac{10}{24} = \frac{2.5}{24}, \text{ and } P(H|S)P(S) = \frac{1}{6} \times \frac{9}{24} = \frac{1.5}{24}$$

to get the Bayesian decision that it was a curveball. Note that it is not necessary to find the *a posteriori* probabilities $P(F|H)$, $P(C|H)$, and $P(S|H)$ explicitly; the joint probabilities suffice.

5. (a) The maximum-likelihood decision rule is indicated by the boldface entries in the likelihood matrix shown below.

Hypothesis	$\mathcal{X} = 3$	$\mathcal{X} = 6$	$\mathcal{X} = 9$	$\mathcal{X} = 12$
H_0 : excellent	0.02	0.08	0.15	0.75
H_1 : good	0.1	0.15	0.6	0.15
H_2 : average	0.2	0.65	0.1	0.05

- (b) $P(\text{excellent student is labeled as good}) = P(\mathcal{X} = 9|H_0) = 0.15$.
 $P(\text{excellent student is labeled as average}) = P(\{\mathcal{X} = 6\} \cup \{\mathcal{X} = 3\}|H_0) = 0.02 + 0.08 = 0.1$.
 $P(\text{average student is labeled as good or excellent}) = P(\{\mathcal{X} = 9\} \cup \{\mathcal{X} = 12\}|H_2) = 0.15$.
- (c) The conditional error probabilities of the maximum-likelihood decision rule are $P(E|H_0) = 0.25$, $P(E|H_1) = 0.4$, $P(E|H_2) = 0.15$. Hence, the error probability is

$$P(E) = P(E|H_0)P(H_0) + P(E|H_1)P(H_1) + P(E|H_2)P(H_2) = 0.05 + 0.22 + .0375 = 0.3075.$$

- (d) The joint probability matrix is as shown below together with the MAP decision rule.

Hypothesis	$\mathcal{X} = 3$	$\mathcal{X} = 6$	$\mathcal{X} = 9$	$\mathcal{X} = 12$
H_0 : excellent	0.004	0.016	0.03	0.15
H_1 : good	0.055	0.0825	0.33	0.0825
H_2 : average	0.05	0.1625	0.025	0.0125

$P(E) = 1 - (0.15 + 0.33 + 0.1625 + 0.055) = 0.3025$ which is slightly smaller than that of the maximum-likelihood rule. But note that students getting D's are classified as good while students getting C's are classified as average. Holy capricious grading complaint, Batman!

- (e) Now the joint probability matrix looks as shown below, and all students are classified as excellent regardless of their grade on the exam!

Hypothesis	$\mathcal{X} = 3$	$\mathcal{X} = 6$	$\mathcal{X} = 9$	$\mathcal{X} = 12$
H_0 : excellent	0.019	0.076	0.1425	0.7125
H_1 : good	0.005	0.0075	0.03	0.0075
H_2 : average	0	0	0	0

Obviously, there is no need for examinations at the Lake Wobegon campus since the results are ignored anyway.

Noncredit exercise: Write a letter to the Governor asking him to suggest to the University that the Lake Wobegon approach be adopted as a cost-cutting measure ...