

ECE 413: Solutions to Problem Set 4

1. $\mathcal{X}$ takes on values 0, 1, 2, 3, 4, 5, 6, 7, 8 with probabilities $p_{\mathcal{X}}(i)$ $\frac{1}{256}, \frac{8}{256}, \frac{28}{256}, \frac{56}{256}, \frac{70}{256}, \frac{56}{256}, \frac{28}{256}, \frac{8}{256}, \frac{1}{256}$ respectively.

(a) $E[\mathcal{X}] = \sum_{i=0}^8 i \cdot p_{\mathcal{X}}(i) = 4$ by tedious calculation, or, more simply, $E[\mathcal{X}] = np = 4$.

(b) $\mathcal{Y}$ takes on values 1, 2, 3, 0 with probabilities $\frac{28}{256}, \frac{8}{256}, \frac{1}{256}$, and $1 - \frac{28+8+1}{256} = \frac{219}{256}$ respectively.

(c) i. $E[\mathcal{Y}] = \sum_{i=0}^3 i \cdot p_{\mathcal{Y}}(i) = \frac{47}{256}$. ii. $E[\mathcal{Y}] = \sum_{i=0}^8 \max\{\mathcal{X} - 5, 0\} \cdot p_{\mathcal{X}}(i) = \frac{47}{256}$

2. Each die has probability $\frac{1}{6}$ of matching your chosen number. Thus, the number of dice that match is a binomial random variable with parameters $(3, \frac{1}{6})$.

(a) (b) You lose \$6 if none of the dice show your chosen number. This occurs with probability $(\frac{5}{6})^3 = \frac{125}{216}$. If one of the dice matches, you win \$6 with probability $3 \cdot \frac{1}{6} \cdot (\frac{5}{6})^2 = \frac{75}{216}$. You win \$12 with probability $3 \cdot \frac{5}{6} \cdot (\frac{1}{6})^2 = \frac{15}{216}$, and you win \$18 with probability $(\frac{1}{6})^3 = \frac{1}{216}$. Rationality test: $125 + 75 + 15 + 1 = 216$.

(b) $E[\mathcal{X}] = (-6)\frac{125}{216} + 6\frac{75}{216} + 12\frac{15}{216} + 18\frac{1}{216} = -\frac{102}{216}$.

(c) If all three dice show different numbers (which has probability $\frac{6 \cdot 5 \cdot 4}{6 \cdot 6 \cdot 6} = \frac{120}{216}$, you win \$1 on the three numbers but lose \$1 on the other three for a net gain of \$0. If two dice show the same number (which has probability $3 \cdot \frac{6 \cdot 1 \cdot 5}{6 \cdot 6 \cdot 6} = \frac{90}{216}$ of occurring) you win \$2 for that number and \$1 for the other number showing, but lose \$1 on the remaining four for a net loss of \$1. If all three dice show the same number (which has probability $6 \cdot \frac{1 \cdot 1 \cdot 1}{6 \cdot 6 \cdot 6} = \frac{6}{216}$ of occurring, you win \$3 for that number, but lose \$1 on the other five for a net loss of \$2. Thus,

$$E[\mathcal{Y}] = (0)\frac{120}{216} + (-1)\frac{90}{216} + (-2)\frac{6}{216} = -\frac{102}{216}.$$

Thus, the average loss is the same regardless of whether you split your bet or not. Worse yet, your wealth will never increase if you split your bet: you will, at best, break even, whereas if you bet your money on one number, there will be a small chance that you will be ahead at some time.

3. (a) $P\{\mathcal{Y} \text{ is even}\} = P\{\mathcal{Y} = 0\} + P\{\mathcal{Y} = 2\} + P\{\mathcal{Y} = 4\} + \dots = \exp(-\lambda) \left[1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots \right] = \exp(-\lambda) \cosh(\lambda)$ by recognizing the series in square brackets.

(b) $\frac{1}{2} [1 + (1 - 2p)^N] = \frac{1}{2} \left[1 + \left(1 - \frac{2\lambda}{N} \right)^N \right] \rightarrow \frac{1}{2} [1 + \exp(-2\lambda)] = \exp(-\lambda) \frac{\exp(\lambda) + \exp(-\lambda)}{2} = \exp(-\lambda) \cosh(\lambda)$.

- (c) The likelihood that we observed $\mathcal{Y} = k$ is $\exp(-\lambda)\lambda^k/k!$. This has derivative $[-\exp(-\lambda)\lambda^k + k\exp(-\lambda)\lambda^{k-1}]/k! = 0$ at $\lambda = k$. Thus, the maximum-likelihood estimate of λ when $\mathcal{Y} = k$ is observed is $\hat{\lambda} = k$. Note that for a binomial random variable, the maximum likelihood estimate of p is k/n , i.e., the maximum-likelihood estimate of np is k , and recall that the Poisson random variable with parameter $\lambda = np$ approximates the binomial random variable. Therefore, our above result is consistent with the previous one.

4. (a) On average, $E[\mathcal{X}] = 105 \times 0.9 = 94.5$ passengers show up for the flight.
- (b) $P\{\mathcal{X} \leq 100\} = 1 - P\{\mathcal{X} > 100\}$
 $= 1 - P\{\mathcal{X} = 101\} - P\{\mathcal{X} = 102\} - P\{\mathcal{X} = 103\} - P\{\mathcal{X} = 104\} - P\{\mathcal{X} = 105\}$
 $= 1 - \binom{105}{101}(0.9)^{101}(0.1)^4 - \binom{105}{102}(0.9)^{102}(0.1)^3 - \binom{105}{103}(0.9)^{103}(0.1)^2 - \binom{105}{104}(0.9)^{104}(0.1)^1 - \binom{105}{105}(0.9)^{105}(0.1)^0$
 $= 1 - \binom{105}{4}(0.9)^{101}(0.1)^4 - \binom{105}{3}(0.9)^{102}(0.1)^3 - \binom{105}{2}(0.9)^{103}(0.1)^2 - \binom{105}{1}(0.9)^{104}(0.1)^1 - \binom{105}{0}(0.9)^{105}(0.1)^0$
 $= 0.9832 \dots$
- (c) We saw earlier that if $\mathcal{X}$ is a binomial random variable with parameters (n, p) , then $\mathcal{Y} = n - \mathcal{X}$ is a binomial random variable with parameters $(n, 1 - p)$.
- (d) $P\{\mathcal{Y} \geq 5\} = 1 - P\{\mathcal{Y} = 0\} - P\{\mathcal{Y} = 1\} - P\{\mathcal{Y} = 2\} - P\{\mathcal{Y} = 3\} - P\{\mathcal{Y} = 4\}$
 $= 1 - \exp(-10.5) \left[1 + \frac{10.5}{1!} + \frac{(10.5)^2}{2!} + \frac{(10.5)^3}{3!} + \frac{(10.5)^4}{4!} \right] = 0.9789 \dots$ which is close enough to 0.9832 ... for gummint work.
5. (a) We are modeling the guesses as independent trials, and we are not allowing for other possibilities such as on some questions, the student can eliminate one or more alternatives and thus improve chances of getting the right answer to $\frac{1}{4}$ or $\frac{1}{3}$ etc.
- (b) The likelihood that we observed $\mathcal{W} = n$ is $\binom{N-K}{n} (0.8)^n (0.2)^{N-K-n}$ for $n = 0, 1, \dots, N - K$.
- (c) For given N and n , the likelihood of part (b) is a function, say $f(K)$ of K . We have that

$$\frac{f(K)}{f(K-1)} = \frac{\binom{N-K}{n} (0.8)^n (0.2)^{N-K-n}}{\binom{N-(K-1)}{n} (0.8)^n (0.2)^{N-(K-1)-n}} = \frac{N - (K-1) - n}{0.2(N - (K-1))} \geq 1 \text{ iff } K \leq N - 1.25n + 1.$$

Thus, the likelihood is maximum when K has value $\hat{K} = \lfloor N - 1.25n + 1 \rfloor$.

- (d) If n is *not* a multiple of 4, then the analysis of part (b) shows that $f(\hat{K})$ is the unique maximum of the likelihood, and the examiner's estimate $\tilde{K} = N - n - \lfloor 0.25n \rfloor$ equals the maximum likelihood estimate. When n is a multiple of 4, then $f(\hat{K}) = f(\hat{K} - 1)$, and thus both $\hat{K}$ and $\hat{K} - 1$ are legitimate maximum likelihood estimates of the unknown quantity. Note that in this case, the examiner's estimate $\tilde{K} = \hat{K} - 1$ and thus the examiner *is* using a maximum likelihood estimate (she is just a tough grader!).

For $N = 100$ and $K = 90$, $\mathcal{W}$ can take on values $0, 1, \dots, 10$. The value of $\mathcal{W}$ most likely to occur is 8. In this case, $\hat{K} = 91$ while $\tilde{K} = 90$ and so the examiner *does* estimate K correctly. On the other hand, if $\mathcal{W} = 4$, the examiner estimates K to be $\tilde{K} = 95$ (erring on the side of caution), and if $\mathcal{W} = 10$, then $\tilde{K} = 88$, ouch!

6. Note that $\mathcal{X}$ can take on values 1, 2, 3, 4, 5, 6 only. Since all are equally likely to be in first place, and there are 5 women in the group of 10, we easily get that $P\{\mathcal{X} = 1\} = \frac{5}{10} = \frac{1}{2}$. Similarly, since there are 5 choices for the man in first place and 5 choices for the woman in second place, we have that $P\{\mathcal{X} = 2\} = \left(\frac{5}{10}\right) \left(\frac{5}{9}\right) = \frac{5}{18}$.

Continuing in this vein, we get that $P\{\mathcal{X} = 3\} = \left(\frac{5}{10}\right) \left(\frac{4}{9}\right) \left(\frac{5}{8}\right) = \frac{10}{72} = \frac{5}{36}$,

$$P\{\mathcal{X} = 4\} = \left(\frac{5}{10}\right) \left(\frac{4}{9}\right) \left(\frac{3}{8}\right) \left(\frac{5}{7}\right) = \frac{5}{84},$$

$$P\{\mathcal{X} = 5\} = \left(\frac{5}{10}\right) \left(\frac{4}{9}\right) \left(\frac{3}{8}\right) \left(\frac{2}{7}\right) \left(\frac{5}{6}\right) = \frac{5}{252},$$

$$P\{\mathcal{X} = 6\} = \left(\frac{5}{10}\right) \left(\frac{4}{9}\right) \left(\frac{3}{8}\right) \left(\frac{2}{7}\right) \left(\frac{1}{6}\right) \left(\frac{5}{5}\right) = \frac{1}{252}.$$

It is useful to check the answers for rationality: is the sum of all the probabilities 1? Yes!