

ECE 413: Solutions to Problem Set 3

1. (a) $\Gamma(1) = \int_0^\infty \exp(-x) dx = -\exp(-x) \Big|_0^\infty = 1.$

(b) For $\alpha > 0$, $\Gamma(\alpha + 1) = \int_0^\infty x^\alpha \exp(-x) dx = x^\alpha [-\exp(-x)] \Big|_0^\infty - \int_0^\infty \alpha x^{\alpha-1} [-\exp(-x)] dx$
 $= 0 + \alpha \int_0^\infty x^{\alpha-1} \exp(-x) dx = \alpha \Gamma(\alpha).$

Hence $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\cdots 2 \cdot 1 \cdot \Gamma(1) = n!.$

(c) Similarly, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha) = \alpha(\alpha-1)\Gamma(\alpha-1) = \cdots = \alpha(\alpha-1)(\alpha-2)\cdots \Gamma(\alpha - \lfloor \alpha \rfloor)$ where $\lfloor \alpha \rfloor$ is the *integer part* of α , and $0 < \alpha - \lfloor \alpha \rfloor < 1.$

(d) Using the substitution $y = \sqrt{2x}$ and $dx/\sqrt{x} = \sqrt{2}dy$, we get

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} \exp(-x) dx = \sqrt{2} \int_0^\infty \exp(-y^2/2) dy = \sqrt{\pi}$$

since from Problem 2(c) of Problem Set 2 we know that the value of the last integral is $\sqrt{\pi/2}.$

2. $P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C))$ by DeMorgan's theorem. Thus,

(a) $B \cap C = \emptyset$ and hence $P(A \cup (B \cap C)) = P(A \cup \emptyset) = P(A) = 1/3.$

(b) $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C))$
 $= 4P(A \cap B \cap C) + 2P(A \cap B \cap C) - P(A \cap B \cap C) = 5P(A \cap B \cap C) = 5/8.$

(c) $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = 1/2 + 1/3 - 0 = 5/6.$ Why?

(d) $(A \cup (B^c \cup C^c)^c)^c = A^c \cap (B^c \cup C^c).$ Hence, $P(A \cup (B^c \cup C^c)^c) = 1 - 0.6 = 0.4.$

3. (a) i. The sample space consists of 5-tuples of the form (B, C, B, B, C) where B and C have the obvious meaning, and 3 of the entrees must be B and 2 must be C . Obviously, the broccolous days can be chosen in $\binom{5}{3} = 10$ ways and hence $|\Omega| = 10.$

ii. $P(\text{broccoli on Monday}) = \binom{4}{2} / \binom{5}{2} = 3/5.$

iii. $P(\text{broccoli on Monday and Friday}) = \binom{3}{2} / \binom{5}{2} = 3/10.$

iv. $P(\text{broccoli on Monday, Wednesday, and Friday}) = 1 / \binom{5}{2} = 1/10.$

(b) We now have 3 independent trials of an experiment.

i. $P(\text{same veg on three days}) = P(AAA) + P(BBB) + P(CCC) = (0.2)^3 + (0.5)^3 + (0.3)^3 = 0.16.$

ii. Let $A^c = B \cup C$. Then, $P(\text{same veg on two days}) = 3[P(AAA^c) + P(BBB^c) + P(CCC^c)]$
 $= 3[(0.2)^2 \times 0.8 + (0.5)^2 \times 0.5 + (0.3)^2 \times 0.7] = 0.66.$ Why the factor 3?

iii. $P(\text{three different}) = 3!P(A)P(B)P(C) = 0.18.$ But why 3!?? Alternatively, we can compute this as $1 - 0.16 - 0.66 = 0.18.$ Why?

4. (a) i. $P(\Omega) = \sum_{n=0}^\infty \frac{(\ln 2)^n}{2(n!)} = \frac{1}{2} \sum_{n=0}^\infty \frac{(\ln 2)^n}{n!} = \frac{1}{2} \exp(\ln 2) = 1.$

ii. $P(n \text{ is even}) = \sum_{n=0}^\infty \frac{(\ln 2)^{2n}}{2(2n)!}.$ Now, $\exp(x) + \exp(-x) = 2 \cosh(x) = 2 \sum_{n=0}^\infty \frac{x^{2n}}{(2n)!}.$ Hence,
 $P(n \text{ is even}) = (1/4)[\exp(\ln 2) + \exp(-\ln 2)] = 5/8.$

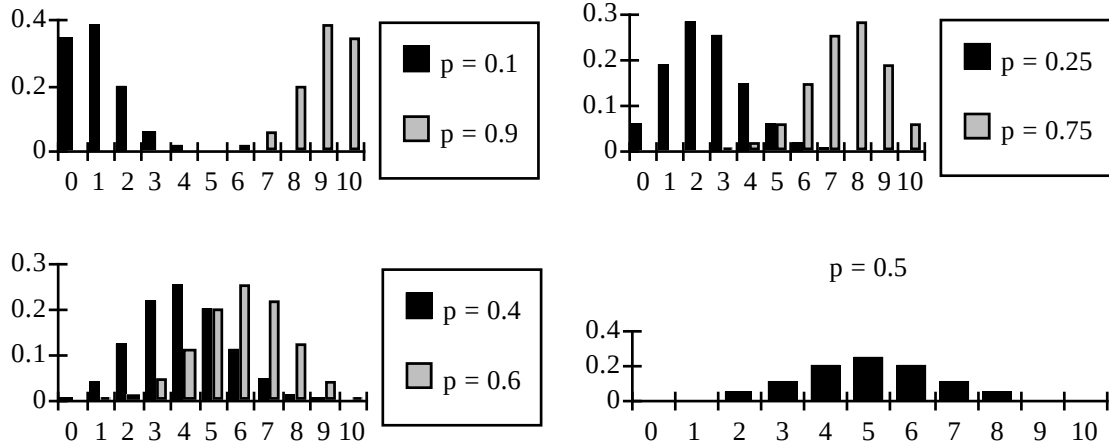
(b) $P(B) = P(H) + P(TTTTH) + P(TTTTTTTH) + \cdots = p + q^4p + q^8p + \cdots = \frac{p}{1 - q^4}.$

$P(C) = P(TH) + P(TTTTH) + P(TTTTTTTH) + \cdots = q[p + q^4p + q^8p + \cdots] = \frac{pq}{1 - q^4}.$

Similarly, we have that $P(T) = \frac{pq^2}{1 - q^4}$ and $P(A) = \frac{pq^3}{1 - q^4}.$ A nicer way of expressing these results is to note that $1 - q^4 = (1 - q)(1 + q + q^2 + q^3)$ and therefore

$P(B) = \frac{1}{1+q+q^2+q^3}, P(C) = \frac{q}{1+q+q^2+q^3}, P(T) = \frac{q^2}{1+q+q^2+q^3}, P(A) = \frac{q^3}{1+q+q^2+q^3}.$
 Since $q < 1$, we have that $P(B) > P(C) > P(T) > P(A)$ which perhaps explains why Alice doesn't live here anymore. Also, quite obviously, $P(B) + P(C) + P(T) + P(A) = 1$.

5. (a) (b) The graphs are as shown below.



(c) The pmfs of X_p and X_{1-p} are “reverses” of each other. For each choice of k , $k = 0, 1, \dots, 10$, and any value of p , $P\{X_p = k\} = P\{X_{1-p} = 10 - k\}$. The reason for this is essentially that X_p counts the number of occurrences on 10 independent trials of an event A of probability p . But, if $X_p = k$, then the *complementary event* A^c (of probability $1 - p$), must have occurred $10 - k$ times, and the number of occurrences of A^c is counted by X_{1-p} . See Problem 6 below which reiterates the same theme.

6. X denotes a binomial random variable with parameters (N, p) . It counts the number of occurrences of an event A of probability p on N independent trials.

(a) Obviously $Y = N - X$ counts the number of occurrences of A^c , an event of probability $1 - p$, on the N independent trials and is thus a binomial random variable with parameters $(N, 1 - p)$.

(b) $P\{X \text{ is even}\} = \binom{N}{0}(1-p)^N + \binom{N}{2}(1-p)^{N-2}p^2 + \binom{N}{4}(1-p)^{N-4}p^4 + \dots$

Now, a minor extension to Problem 1(d) of Problem Set 2 shows that

$(x+y)^N + (x-y)^N = 2 \left[\binom{N}{0}x^N + \binom{N}{2}x^{N-2}y^2 + \dots \right]$, and so, setting $x = 1 - p$, $y = p$, we get

$P\{X \text{ is even}\} = \frac{1}{2} [(1-p+p)^N + (1-p-p)^N] = \frac{1}{2} [1 + (1-2p)^N]$. Notice that the probability is $1/2$ for $p = 1/2$.