

ECE 413: Solutions to Problem Set 2

1. (a) By definition, $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$.

In the binomial theorem $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$, set $x = 1$ to get $\sum_{k=0}^n \binom{n}{k} = 2^n$.

(b) $\binom{n}{k}$ is the number of events of size k defined on a sample space of size n . Since there is a one-to-one correspondence between the events of size k and their *complements* of size $n-k$, we get $\binom{n}{k} = \binom{n}{n-k}$. $\sum_{k=0}^n \binom{n}{k}$ counts the events in order of size. Since the total number of events is 2^n , the sum must equal 2^n . Note that no formula was harmed in the derivation of this result.

(c) $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots(k-1)} \times \frac{n-k+1}{k} = \frac{n-k+1}{k} \binom{n}{k-1}$.

Thus, we get that $\binom{n}{k} \geq \binom{n}{k-1}$ if $n-k+1 \geq k$, that is, if $k \leq (n+1)/2$. Thus, $\binom{n}{k}$ increases with k until $k > (n+1)/2$ from which point it decreases with increasing k . If $k = (n+1)/2$ (which can only occur when n is odd), there are twin peaks $\binom{n}{(n-1)/2} = \binom{n}{(n+1)/2}$. If n is even, then there is a unique maximum at $\binom{n}{n/2}$.

(d) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \cdots$

$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \binom{n}{4}x^4 - \cdots$ from which we get that

$(1+x)^n + (1-x)^n = 2 \left[\binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \cdots \right]$. Set $x = 1$ and note that the left side has value 2^n while the right side is twice the number of sets with an even number of elements. More simply, set $x = 1$ in $(1-x)^n$ to get

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \Rightarrow \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}.$$

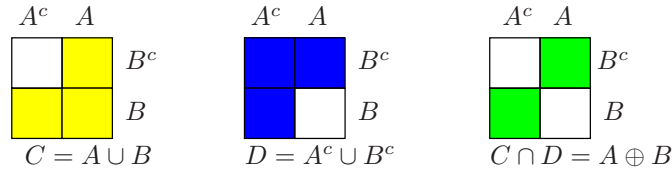
2. (a) By the chain rule, $\frac{d}{dx} \exp(-x^2/2) = \exp(-x^2/2)(-2x/2) = -x \exp(-x^2/2)$.

(b) $\int_0^\infty x \exp(-x^2/2) dx = -\exp(-x^2/2) \Big|_0^\infty = 1$.

(c) $\int_0^\infty \int_0^\infty \exp(-[x^2+y^2]/2) dx dy = \int_{r=0}^\infty \int_{\theta=0}^{\pi/2} \exp(r^2/2) r d\theta dr = \frac{\pi}{2} \int_{r=0}^\infty r \exp(-r^2/2) dr = \frac{\pi}{2}$.

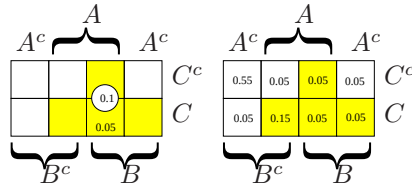
3. (a) Since there are 31 *non-empty* subsets of the set of 5 flavors (you wouldn't want to buy unflavored ice-cream, would you?), the manufacturer can create and sell 31 specialty flavors.

(b) The Karnaugh map shown on the next page shows various events of interest.



We are given that $P(C) = P(A \cup B) = 0.6$ and $P(D) = P(A^c \cup B^c) = 0.8$. Now, note that $C \cup D = \Omega$ while $C \cap D = (A \cup B) \cap (A^c \cup B^c) = (A \cap B^c) \cup (A^c \cap B) = A \oplus B$. Consequently, $P(C \cup D) = P(\Omega) = 1 = P(C) + P(D) - P(A \oplus B) = 0.6 + 0.8 - P(A \oplus B)$ from which we get that $P(A \oplus B) = 0.4$.

4. (a) Since there are $2^7 = 128$ outcomes with $\text{LSB} = 1$, we have that $P(A) = \frac{128}{256} = \frac{1}{2}$.
- (b) There are $\binom{8}{5} = \binom{8}{3} = 56$ outcomes in B and hence $P(B) = \frac{56}{256} = \frac{7}{32}$.
- (c) When the event $A \cap B$ occurs, we know that the 7 more significant bits have exactly 4 ONEs and 3 ZEROes. Hence $|A \cap B| = \binom{7}{4} = \binom{7}{3} = 35$. Thus, $P(A \cap B) = \frac{35}{256}$ and
- $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{128}{256} + \frac{56}{256} - \frac{35}{256} = \frac{149}{256}, \text{ while}$$
- $$P(A \oplus B) = P(A) + P(B) - 2 \cdot P(A \cap B) = \frac{128}{256} + \frac{56}{256} - 2 \cdot \frac{35}{256} = \frac{114}{256} = \frac{57}{128}.$$
- (d) $P(C) = \binom{8}{4} / 256 = \frac{70}{256} = \frac{35}{128}$.
5. (a) There are $\binom{6}{3} = 20$ possible short lists.
- (b) i. $P(\text{Beth is on the short list}) = \binom{5}{2} / 20 = \frac{1}{2}$.
- ii. $P(\text{Chuck is on the short list}) = \binom{5}{2} / 20 = \frac{1}{2}$ also.
- iii. $P(\text{Beth and Chuck both are on the short list}) = \binom{4}{1} / 20 = \frac{1}{5}$.
- iv. $P(\text{Chuck and two women are on the short list}) = \binom{3}{2} / 20 = \frac{3}{20}$.
6. (a) The Karnaugh map is as shown in the left hand figure below, with some probabilities marked on it. Note that the shaded region is the event $(A \cap B) \cup (B \cap C) \cup (A \cap C)$.



- (b) Since $A \cap B$ is the disjoint union of $A \cap B \cap C$ and $A \cap B \cap C^c$, we get that $P(A \cap B) = 0.1 = P(A \cap B \cap C) + P(A \cap B \cap C^c) = 0.05 + P(A \cap B \cap C^c)$ giving that $P(A \cap B \cap C^c) = 0.05$. Since $P(AB \cup BC \cup AC) = 0.3 = P(AB) + P(AB^cC) + P(A^cBC)$ while $P(AC) = P(ABC) + P(AB^cC) = 2P(BC) = P(ABC) + P(A^cBC)$, we readily obtain that $P(AB^cC) = 0.15$ and $P(A^cBC) = 0.05$. Since $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$ we have that $P(A^cB^c) = 1 - P(A \cup B) = 0.4$. Since $P(A^cB^cC) = P(C) - P(BC) - P(AB^cC) = 0.05$, we get that $P(\text{cereal snaps, crackles, and pops}) = P(A^c \cap B^c \cap C) = 0.55$. See the figure on the right.
- (c) $P(\text{the sample fails only the snap test}) = P(AB^cC^c) = 0.05$.
 $P(\text{the sample fails only the crackle test}) = P(A^cBC^c) = 0.05$.
 $P(\text{the sample fails only the pop test}) = P(A^cB^cC) = 0.05$.