

ECE 413: Solutions to Problem Set 1

1. (a) Let $n > 0$ be an integer. Then,

$$(1-x)[1+x+x^2+\cdots+x^{n-1}] = [1+x+x^2+\cdots+x^{n-1}] - [x+x^2+\cdots+x^n] = 1-x^n.$$

If $x \neq 1$, divide both sides by $(1-x)$ to get $1+x+x^2+\cdots+x^{n-1} = \frac{1-x^n}{1-x}$.

- (b) Since $|x| < 1$, $1+x+x^2+\cdots = \lim_{n \rightarrow \infty} 1+x+x^2+\cdots+x^{n-1} = \lim_{n \rightarrow \infty} \frac{1-x^n}{1-x} = \frac{1}{1-x}$.
- (c) The 0-th derivative $f^{(0)}(x)$ is just $f(x) = (1+x)^n$ itself. The first derivative is $f^{(1)}(x) = n(1+x)^{n-1}$, the second is $f^{(2)}(x) = n(n-1)(1+x)^{n-2}$, and so on. The k -th derivative is $f^{(k)}(x) = n(n-1)\cdots(n-k+1)(1+x)^{n-k}$, and finally the n -th derivative is $f^{(n)}(x) = n(n-1)\cdots 2 \cdot 1 = n!$ which is a constant. Hence, the first $n+1$ terms of the Maclaurin series for $f(x)$ are

$$f(x) = (1+x)^n \approx \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^n \frac{n(n-1)\cdots(n-k+1)}{k!} x^k.$$

- (d) Since $f^{(n)}(x)$ is a constant, $f^{(k)}(x) = 0$ for all $k > n$. Thus, the above is the complete Maclaurin series for $f(x)$; there is *no* approximation.
- (e) $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots 2 \cdot 1}{((n-k)(n-k-1)\cdots 2 \cdot 1)k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ and so the results match.
- (f) Using the chain rule,

$$\begin{aligned} g^{(k)}(x) &= (-n)(-n-1)(-n-2)\cdots(-n-k+1)(1-x)^{-n-k}(-1)^k \\ &= n(n+1)(n+2)\cdots(n+k-1)(1-x)^{-n-k} \end{aligned}$$

Thus, $g^{(k)}(0) = n(n+1)(n+2)\cdots(n+k-1) \neq 0$ for all integers $k \geq 0$. Hence, the Maclaurin series for $g(x)$ contains terms of all degrees, and we have that

$$g(x) = (1-x)^{-n} = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{n(n+1)\cdots(n+k-1)}{k!} x^k.$$

- (g) For $n = 1$, $n(n+1)\cdots(n+k-1) = k!$ while for $n = 2$, $n(n+1)\cdots(n+k-1) = (k+1)!$. Hence,

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k = 1+x+x^2+\cdots \quad \text{and} \quad (1-x)^{-2} = \sum_{k=0}^{\infty} (k+1)x^k = 1+2x+3x^2+\cdots.$$

- (h) The k -th derivative of $h(x) = (1+x)^a$ where a is a real number and not necessarily an integer is $h^{(k)}(x) = a(a-1)\cdots(a-k+1)(1+x)^{a-k}$ where the exponent cannot equal 0 when a is not an integer. Thus, the Maclaurin series has terms of all degrees and we have that

$$h(x) = (1+x)^a = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!} x^k.$$

2. (a) Since $\sin x = x - x^3/3! + x^5/5! - \cdots \approx x - x^3/6$ for small x , we have

$$\frac{1}{\sin^2 x} - \frac{1}{x^2} \approx \frac{1}{x^2} \left[\frac{1}{(1-x^2/6)^2} - 1 \right] = \frac{1}{x^2} \left[1 + 2\left(\frac{x^2}{6}\right) + 3\left(\frac{x^2}{6}\right)^2 + \cdots - 1 \right] = \frac{1}{3} + \frac{x^2}{12} + \cdots$$

and so the limiting value as $x \rightarrow 0$ is $1/3$.

- (b) $x^n \exp(-ax)$ has maximum value $\left(\frac{n}{a}\right)^n \exp(-n)$ at $x = n/a$. Here, $n = 25$ and $a = \ln(1.00001)$ giving a maximum value of $0.123365 \dots \times 10^{150}$ at $x = 2500012.5 \dots$

3. (a) $\int_{-1}^2 |x| dx = \int_{-1}^0 -x dx + \int_0^2 x dx = \left. \frac{-x^2}{2} \right|_{-1}^0 + \left. \frac{x^2}{2} \right|_0^2 = \frac{1}{2} + 2 = 2.5.$

The substitution $y = 1 - x$ gives

$$\int_{-2}^1 x(1-x)^{19} dx = \int_3^0 (1-y)y^{19}(-dy) = \int_0^3 y^{19} - y^{20} dy = \frac{3^{20}}{20} - \frac{3^{21}}{21} = -\frac{13 \times 3^{20}}{140}.$$

(b) No nonnegative function $f(x)$ can satisfy $\int_{-2}^1 f(x) dx < 0$. The comparison test says that if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$. Now let $g(x) = 0$. More simply, if the curve $f(x)$ lies above the x axis in an interval $[a, b]$, the area under the curve between a and b cannot be negative.

(c) i. False: the chain rule gives $\frac{d}{dx} f(-x) = -g(-x)$.

ii. False: the chain rule gives $\frac{d}{dx} f(x^2) = g(x^2) \cdot 2x$.

iii. True: according to the chain rule.

iv. False: the chain rule gives $\frac{d}{dx} \exp(f(x^2)) = \exp(f(x^2))g(x^2) \cdot 2x$.

v. False: the antiderivative of $g(-x)$ is $-f(-x)$, (cf i. above.)

vi. False: the antiderivative of $g(x^2)$ need not be related to $f(x^2)$ at all.

(d) i. $I = \int_{-1}^1 \frac{2}{1+x^2} dx = 2 \cdot \arctan(x) \Big|_{-1}^1 = 2 \cdot (\pi/4 - (-\pi/4)) = \pi.$

ii. The substitution $y = 1/x$ changes the integrand to $\frac{2}{1+1/y^2}(-1/y^2)dy = \frac{-2}{1+y^2} dy$ which is the integrand for J . However, as x varies from -1 to 0 , y varies from -1 to $-\infty$. Similarly, as x varies from 0 to 1 , y varies from ∞ to 1 . Therefore, we get

$$\int_{-1}^1 \frac{2}{1+x^2} dx = \int_{-1}^{-\infty} \frac{-2}{1+y^2} dy + \int_{\infty}^1 \frac{-2}{1+y^2} dy = \int_{-\infty}^{-1} \frac{2}{1+y^2} dy + \int_1^{\infty} \frac{2}{1+y^2} dy = \pi.$$

Thus, the substitution $y = 1/x$ does not change I into J as asserted in the problem statement.

iii. No need to re-write the math texts: $\pi \neq 0$.

4. (a) $f(x, y) = \max(x, y)$ takes on values x and y in the regions indicated in the figure below. Hence,

$$\int_{y=0}^1 \int_{x=0}^2 f(x, y) dx dy = \int_{y=0}^1 \int_{x=0}^y y dx dy + \int_{y=0}^1 \int_{x=y}^2 x dx dy = \int_{y=0}^1 y^2 + 2 - y^2/2 dy = \frac{13}{6}.$$

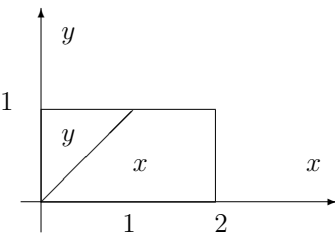


Figure 1: Figure for Problem 4(a)

(b) The integral is over the exterior of a circle of radius 2. Thus, a change to polar coordinates gives

$$\iint_{x^2+y^2>4} (x^2+y^2)^{-2} dx dy = \int_{r=2}^{\infty} \int_{\theta=0}^{2\pi} \frac{1}{r^4} r d\theta dr = 2\pi \left. \frac{-1}{2r^2} \right|_{r=2}^{\infty} = \frac{\pi}{4}.$$