University of Illinois Spring 2005

ECE 413: Solutions to the Final Examination

1. (**a**) A and B are two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.

TRUE $P(A \cup B) \geq \max\{P(A), P(B)\}$
FALSE $P(A \cap B) \geq \min\{P(A), P(B)\}$ $(P(A \cap B) \leq \ldots$, not $\geq \ldots)$
FALSE $P(A|B) + P(A|B^c) = 1$.
TRUE $P(A|B)P(B) + P(A^c|B^c)P(B^c) = 1 - P(A \oplus B)$.
FALSE $P(A|B)P(B) + P(A^c|B)P(B) = P(A)$. (true if RHS were $P(B)$)
TRUE If $P(A) = P(B)$, then $P(A|B) = P(B|A)$.
FALSE If $P(A|B) = P(B|A)$, then $P(A) = P(B)$. Conclusion does not hold if $P(A|B) = P(B|A) = 0$.
TRUE If $P(A|B) = P(A)$, then $P(B^c|A) = 1 - P(B)$. A and B are independent events

(**b**) $\mathcal{X}$ and $\mathcal{Y}$ are random variables such that $\mathsf{var}(\mathcal{X}) = \mathsf{var}(\mathcal{Y}) = \sigma^2 < \infty$.
$\mathsf{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = 4\sigma^2 + 9\sigma^2 + 12 \cdot \mathsf{cov}(\mathcal{X}, \mathcal{Y}) = \mathsf{var}(3\mathcal{X} - 2\mathcal{Y} + 1) = 9\sigma^2 + 4\sigma^2 - 12 \cdot \mathsf{cov}(\mathcal{X}, \mathcal{Y})$ implies that $\mathsf{cov}(\mathcal{X}, \mathcal{Y}) = 0$.

TRUE $\mathcal{X}$ and $\mathcal{Y}$ are *uncorrelated* random variables.
FALSE $\mathcal{X}$ and $\mathcal{Y}$ are *independent* random variables.
TRUE $\mathsf{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = \mathsf{var}(2\mathcal{X} - 3\mathcal{Y} + 1)$.
TRUE $\mathsf{cov}(\mathcal{X} + \mathcal{Y}, \mathcal{X} - \mathcal{Y}) = 0$.

2. (**a**) Let A_n denote the event that the mailman is *not* bitten on the n-th day. Then, $\{\mathcal{X} = n\} = A_n^c \cap A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1$ and $\{\mathcal{X} > n\} = A_n \cap A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1$. We are given that $P\{A_n^c | A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1\} = \dfrac{1}{n+1}$ and hence $P\{A_n | A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1\} = \dfrac{n}{n+1}$. Note also that $P\{A_1\} = P\{A_1^c\} = \dfrac{1}{2}$. Therefore,

$$\begin{aligned}
P\{\mathcal{X} > n\} &= P\{A_n \cap A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1\} \\
&= P\{A_n | A_{n-1} \cap \cdots \cap A_1\} P\{A_{n-1} | A_{n-2} \cap \cdots \cap A_1\} \cdots P\{A_2 | A_1\} P\{A_1\} \\
&= \frac{n}{n+1} \times \frac{n-1}{n} \times \cdots \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{n+1}.
\end{aligned}$$

Hence $p_{\mathcal{X}}(n) = P\{\mathcal{X} = n\} = P\{\mathcal{X} > n-1\} - P\{\mathcal{X} > n\} = \dfrac{1}{n} - \dfrac{1}{n+1} = \dfrac{1}{n(n+1)}$, $n = 1, 2, \ldots$.

(**b**) $\mathsf{E}[\mathcal{X}] = \displaystyle\sum_{n=1}^{\infty} n \cdot P\{\mathcal{X} = n\} = \sum_{n=1}^{\infty} n \cdot \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$ from the fact that the harmonic series diverges. Alternatively, since $\mathcal{X}$ takes on positive integer values, we have from Problem 4(a) of Problem Set 8 that $\mathsf{E}[\mathcal{X}] = \displaystyle\sum_{n=0}^{\infty} P\{\mathcal{X} > n\} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \infty$.

3. The component lifetimes are exponential random variables with parameter λ. Thus, the probability that a component fails before time T is $1 - \exp(-\lambda T)$. Since the failures are independent, we have that $P\{\mathcal{X} \leq T\} = [1 - \exp(-\lambda T)]^3$ while $P\{\mathcal{X} > T\} = 3 \cdot \exp(-\lambda T) - 3 \cdot \exp(-2\lambda T) + \exp(-3\lambda T)$. We can thus write

$$\mathsf{E}[\mathcal{X}] = \int_0^{\infty} P\{\mathcal{X} > T\}\, dT = \int_0^{\infty} 3\exp(-\lambda T) - 3\exp(-2\lambda T) + \exp(-3\lambda T)\, dT = \frac{3}{\lambda} - \frac{3}{2\lambda} + \frac{1}{3\lambda} = \frac{11}{6\lambda}.$$

4. For $-1 < u < 1$, the likelihood ratio is $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)} = \dfrac{1 - |u|}{1/2} = 2 - 2|u|$.

(**a**) When $\mathcal{X} = u$ is the observation, the *maximum-likelihood* decision rule decides in favor of H_1 if $\Lambda(u) > 1$. Hence $\Gamma_1 = \left\{u \ : \ |u| < \frac{1}{2}\right\}$ and $\Gamma_0 = \left\{u \ : \ \frac{1}{2} < |u| < 1\right\}$.

(**b**) $P_{\text{FA}} = \displaystyle\int_{\Gamma_1} f_0(u)\, du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} du = \frac{1}{2}$. $P_{\text{MD}} = \displaystyle\int_{\Gamma_0} f_1(u)\, du = 2\int_{\frac{1}{2}}^{1} (1-u)\, du = 2\left(u - \frac{u^2}{2}\right)\Bigg|_{\frac{1}{2}}^{1} = \frac{1}{4}$.

(**c**) When $\mathcal{X} = u$ is the observation, the MAP decision rule decides in favor of H_1 if $\Lambda(u) > \pi_0/\pi_1$. Hence, $\Gamma_1 = \{u \,:\, |u| < \frac{5}{6}\}$ and $\Gamma_0 = \{u \,:\, \frac{5}{6} < |u| < 1\}$ for the MAP decision rule. We get $P_{\text{FA}} = \int_{\Gamma_1} f_0(u)\,du = \int_{-\frac{5}{6}}^{\frac{5}{6}} \frac{1}{2} du = \frac{5}{6}$. $P_{\text{MD}} = \int_{\Gamma_0} f_1(u)\,du = 2\int_{\frac{5}{6}}^{1} (1-u)\,du = 2\left(u - \frac{u^2}{2}\right)\Big|_{\frac{5}{6}}^{1} = \frac{1}{36}$.
Hence, $\bar{P}_e = \pi_0 \cdot P_{\text{FA}} + \pi_1 \cdot P_{\text{MD}} = \frac{1}{4} \times \frac{5}{6} + \frac{3}{4} \times \frac{1}{36} = \frac{5}{24} + \frac{1}{48} = \frac{11}{48}$.

(**d**) The MAP decision rule compares the likelihood ratio $\Lambda(u) = 2 - 2|u|$ and decides in favor of H_0 if $\Lambda(u) < \pi_0/\pi_1$. Since $\Lambda(u)$ takes on values in $(0, 2]$, it will never exceed π_0/π_1 if $\pi_0 > \frac{2}{3}$. Thus, the MAP decision rule always decides in favor of H_0 if $\pi_0 > 2/3$, and achieves average error probability $\bar{P}_e = \pi_1 < 1/3$.
On the other hand, there is *no* value of $\pi_0, 0 < \pi_0 < 1$ for which the MAP rule *always* decide in favor of H_1, because no matter how large π_1 is, the ratio $\pi_0/\pi_1 > 0$. Thus, for *some* values of $\mathcal{X}$, the likelihood ratio will be smaller than the threshold, and the decision will be in favor of H_0.

5. (**a**) The inter-arrival time in a Poisson process with arrival rate λ (time between two successive chalk breaks on this instance) is an exponential random variable with parameter λ. Hence, the average length of time between successive chalk-breaks is the mean of this exponential random variable, which is $\lambda^{-1} = 10$ minutes.

(**b**) The number of times that the professor breaks the chalk during a 50 minute lecture is a Poisson random variable $\mathcal{N}(0, 50]$ with parameter $\lambda \times 50 = 5$ and mean value $\mathsf{E}[\mathcal{N}(0, 50]] = 5$.

(**c**) From part (b), we get that $P\{\mathcal{N}(0, 50] = 6\} = \frac{5^6}{6!}\exp(-5)$. Now, for $0 \leq k \leq 6$,

$$\begin{aligned}
P\{\{\mathcal{N}(0,25] = k\} \mid \{\mathcal{N}(0,50] = 6\}\} &= \frac{P\{\{\mathcal{N}(0,25] = k\} \cap \{\mathcal{N}(0,50] = 6\}\}}{P\{\mathcal{N}(0,50] = 6\}\}} \\
&= \frac{P\{\{\mathcal{N}(0,25] = k\} \cap \{\mathcal{N}(25,50] = 6-k\}\}}{P\{\mathcal{N}(0,50] = 6\}\}} \\
&= \frac{P\{\mathcal{N}(0,25] = k\}P\{\mathcal{N}(25,50] = 6-k\}\}}{P\{\mathcal{N}(0,50] = 6\}\}} \\
&= \frac{\frac{(2.5)^k}{k!}\exp(-2.5) \times \frac{(2.5)^{6-k}}{(6-k)!}\exp(-2.5)}{\frac{5^6}{6!}\exp(-5)} \\
&= \binom{6}{k}\left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{6-k}
\end{aligned}$$

Thus, the *conditional* pmf of $\mathcal{N}(0, 25]$ given that $\{\mathcal{N}(0, 50] = 6\}$ is a *binomial* pmf with parameters $(6, 1/2)$ and hence the expected value is 3.

6. (**a**) $f_{\mathcal{X}}(u) = \int_{-\infty}^{\infty} f_{\mathcal{X},\mathcal{Y}}(u,v)\,dv = \int_0^1 u + v\,dv = uv + \frac{v^2}{2}\Big|_0^1 = u + \frac{1}{2}$, for $0 < u < 1$, and $f_{\mathcal{X}}(u) = 0$ otherwise.

(**b**) It is obvious from the symmetry of the problem that $f_{\mathcal{Y}}(v) = v + 0.5$ for $0 < v < 1$. We readily see that $f_{\mathcal{X},\mathcal{Y}}(u,v) \neq f_{\mathcal{X}}(u)f_{\mathcal{Y}}(v)$ and hence $\mathcal{X}$ and $\mathcal{Y}$ are *dependent* random variables

(**c**) $f_{\mathcal{Y}|\mathcal{X}}\left(v \,\Big|\, \frac{1}{3}\right) = \frac{f_{\mathcal{X},\mathcal{Y}}\left(\frac{1}{3}, v\right)}{f_{\mathcal{X}}\left(\frac{1}{3}\right)} = \frac{\frac{1}{3} + v}{\frac{5}{6}} = \frac{2}{5} + \frac{6}{5}v$ for $0 < v < 1$, and $f_{\mathcal{Y}|\mathcal{X}}\left(v \,\Big|\, \frac{1}{3}\right) = 0$ otherwise.

7. The impulse response is a damped oscillation if the roots of $s^2 + \mathcal{A}s + \mathcal{B}$ are complex numbers, that is, if $\mathcal{A}^2 < 4\mathcal{B}$. We have $P\{\mathcal{A}^2 < 4\mathcal{B}\} = \int_{u=0}^{1}\int_{v=u^2/4}^{1} 1\,dv\,du = \int_0^1 1 - \frac{u^2}{4}\,du = u - \frac{u^3}{12}\Big|_0^1 = \frac{11}{12}$.

8. $\mathcal{Z} = \mathcal{Y} - \mathcal{X}$ takes on values in $(0, 1)$. We readily find that for $0 < \alpha < 1$,
$1 - F_{\mathcal{Z}}(\alpha) = P\{\mathcal{Z} > \alpha\} = P\{\mathcal{Y} - \mathcal{X} > \alpha\} = P\{\mathcal{Y} > \mathcal{X} + \alpha\} = 2 \times ((1-\alpha)^2/2) = (1-\alpha)^2$.
Differentiating, we get $f_{\mathcal{Z}}(\alpha) = 2(1-\alpha)$ for $0 < \alpha < 1$, and $f_{\mathcal{Z}}(\alpha) = 0$ otherwise.