University of Illinois Spring 2005

ECE 413: Final Examination Friday May 6, 2005

1. (**a**) A and B are two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.
Mark TRUE or FALSE for each question below.

TRUE	FALSE	
□	□	$P(A \cup B) \geq \max\{P(A), P(B)\}$
□	□	$P(A \cap B) \geq \min\{P(A), P(B)\}$
□	□	$P(A\|B) + P(A\|B^c) = 1$.
□	□	$P(A\|B)P(B) + P(A^c\|B^c)P(B^c) = 1 - P(A \oplus B)$.
□	□	$P(A\|B)P(B) + P(A^c\|B)P(B) = P(A)$.
□	□	If $P(A) = P(B)$, then $P(A\|B) = P(B\|A)$.
□	□	If $P(A\|B) = P(B\|A)$, then $P(A) = P(B)$.
□	□	If $P(A\|B) = P(A)$, then $P(B^c\|A) = 1 - P(B)$.

(**b**) $\mathcal{X}$ and $\mathcal{Y}$ are random variables such that $\mathsf{var}(\mathcal{X}) = \mathsf{var}(\mathcal{Y}) = \sigma^2 < \infty$.
Suppose that $\mathsf{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = \mathsf{var}(3\mathcal{X} - 2\mathcal{Y} + 1)$.
Mark TRUE or FALSE for each of the following statements.

TRUE	FALSE	
□	□	$\mathcal{X}$ and $\mathcal{Y}$ are *uncorrelated* random variables.
□	□	$\mathcal{X}$ and $\mathcal{Y}$ are *independent* random variables.
□	□	$\mathsf{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = \mathsf{var}(2\mathcal{X} - 3\mathcal{Y} + 1)$.
□	□	$\mathsf{cov}(\mathcal{X} + \mathcal{Y}, \mathcal{X} - \mathcal{Y}) = 0$.

2. A mailman starting a new route estimates that the probability that he is bitten by a dog on the very first day is $\frac{1}{2}$. As each day passes without the mailman being bitten, he grows increasingly wary (and he becomes more cognisant of the locations of the dogs on his route). The *conditional* probability that the mailman is bitten on the n-th day, given that he has not been bitten on days 1 through $n-1$, is $\frac{1}{n+1}$.

Let $\mathcal{X}$ denote the day on which the mailman is first bitten by a dog (at which time he goes postal, shoots the dog, and is transferred to a new route.)

(**a**) What is the pmf of $\mathcal{X}$? Hint: first find $P\{\mathcal{X} > n\}$ for $n = 1, 2, \ldots$

(**b**) (This one is for dog lovers everywhere.) What is the expected value of $\mathcal{X}$?

3. A system contains three components that fail independently of each other. *The system fails when all three components have failed.* Suppose that each component has *constant* hazard rate $\lambda > 0$. What is the expected value of the *system lifetime* $\mathcal{X}$?

4. A radio-frequency signal is either a radar echo (hypothesis H_1) or ambient noise (hypothesis H_0). The *phase* of the signal is modeled as $\pi\mathcal{X}$ where $\mathcal{X}$ is a continuous random variable.

- When H_0 is true, $\mathcal{X}$ is *uniformly distributed* on $(-1, +1)$.
- When H_1 is true, $\mathcal{X}$ has pdf $f(u) = \begin{cases} 1 - |u|, & -1 < u < +1, \\ 0, & \text{elsewhere.} \end{cases}$

The radar receiver measures $\mathcal{X}$ and must decide whether the signal is an echo or noise.

(**a**) Suppose that the *maximum-likelihood* decision rule is being used. What value(s) of $\mathcal{X}$ result in a decision in favor of H_1?

(**b**) Find the *false alarm* probability P_{FA} and the *missed detection* or *false dismissal* probability P_{MD} of the maximum-likelihood decision rule.

(**c**) Now suppose that $P(\mathsf{H}_0) = \pi_0 = \frac{1}{4}$, $P(\mathsf{H}_1) = \pi_1 = \frac{3}{4}$. What is the *average* error probability $\bar{P}_e$ of the maximum *a posteriori* probability (MAP) (that is, minimum-error-probability or Bayesian) decision rule?

(**d**) For what values, if any, of $\pi_0, 0 < \pi_0 < 1$ does the MAP rule *always* decide in favor of H_0 regardless of the value of $\mathcal{X}$?
For what values, if any, of $\pi_0, 0 < \pi_0 < 1$ does the MAP rule *always* decide in favor of H_1 regardless of the value of $\mathcal{X}$??

5. A professor breaks the chalk piece with which he is writing on the blackboard at random times that can be modeled as arrivals in a Poisson process with arrival rate $\lambda = 0.1$ per minute.

(**a**) What is the expected length of time between two successive chalk breaks?

(**b**) What is the average number of times that the professor breaks the chalk during a 50 minute lecture?

(**c**) *Given* that the professor broke 6 chalk pieces in 50 minutes, what is the average number of pieces he broke in the first 25 minutes?

6. The jointly continuous random variables $\mathcal{X}$ and $\mathcal{Y}$ have joint pdf given by

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} u+v, & 0<u<1, 0<v<1, \\ 0, & \text{elsewhere.} \end{cases}$$

(**a**) Find the marginal pdf $f_{\mathcal{X}}(u)$ of the random variable $\mathcal{X}$. Be sure to specify the value of $f_{\mathcal{X}}(u)$ for all $u, -\infty < u < \infty$.

(**b**) Are $\mathcal{X}$ and $\mathcal{Y}$ independent random variables?

(**c**) Find the *conditional* pdf of $\mathcal{Y}$ given that $\mathcal{X} = \frac{1}{3}$. Be sure to specify the value of $f_{\mathcal{Y}|\mathcal{X}}(v|\frac{1}{3})$ for all $v, -\infty < v < \infty$.

7. A linear time-invariant system has transfer function

$$H(s) = \frac{1}{s^2 + \mathcal{A}s + \mathcal{B}}$$

where $\mathcal{A}$ and $\mathcal{B}$ are independent random variables uniformly distributed on $(0,1)$. What is the probability that the system impulse response is a decaying oscillation? The forgetful are reminded that the impulse response is a decaying oscillation if the transfer function (regarded as a function of the complex variable s) has poles in the left half-plane that are not on the real axis.

8. The random point $(\mathcal{X},\mathcal{Y})$ is uniformly distributed on the region $\{(u,v) : 0 < u < v < 1\}$.
Find the pdf of the random variable $\mathcal{Z} = \mathcal{Y} - \mathcal{X}$. Be sure to specify the value of $f_{\mathcal{Z}}(\alpha)$ for all $\alpha, -\infty < \alpha < \infty$.