University of Illinois Spring 2005

ECE 413: Solutions to Hour Exam II

1. (**a**)

$$\begin{aligned} P(D) &= P((A\cap B)\cup(B\cap C)\cup(A\cap C)) \\ &= P(A\cap B)+P(A\cap C)+P(B\cap C)-2P(A\cap B\cap C) \\ &= P(A)P(B)+P(A)P(C)+P(B)P(C)-2P(A)P(B)P(C) \quad \text{by independence} \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)+\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)-2\times\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) \\ &= \frac{13}{24}. \end{aligned}$$

(**b**) $P(A^c|D) = \dfrac{P(A^c\cap D)}{P(D)} = \dfrac{P(A^c\cap B\cap C)}{13/24} = \dfrac{(2/3)\times(1/2)\times(3/4)}{13/24} = \dfrac{6}{13}.$

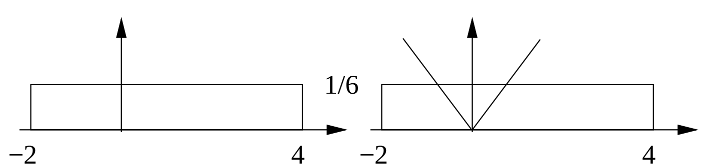
2. (**a**) $\Lambda(k) = \dfrac{p_1(k)}{p_0(k)} = \dfrac{(\ln 27)^k\exp(-\ln 27)/k!}{(\ln 9)^k\exp(-\ln 9)/k!} = \left(\dfrac{\ln 27}{\ln 9}\right)^k \dfrac{9}{27} = \left(\dfrac{3\ln 3}{2\ln 3}\right)^k \dfrac{1}{3} = \dfrac{3^{k-1}}{2^k}.$

(**b**) Evaluating $\Lambda(k)$ for $k = 0,1,2,\ldots$, we get that the decision is in favor of H_1 if $\mathcal{X}$ takes on values $k \geq 3$.

(**c**) $P_{\text{FA}} = \displaystyle\sum_{k=3}^{\infty}\frac{(\ln 9)^k}{k!}\exp(-\ln 9) = 1-\frac{1}{9}\sum_{k=0}^{2}\frac{(\ln 9)^k}{k!} = 1-\frac{1}{9}\left[1+\ln 9+\frac{(\ln 9)^2}{2!}\right].$

$P_{\text{MD}} = \displaystyle\sum_{k=0}^{2}\frac{(\ln 27)^k}{k!}\exp(-\ln 27) = \frac{1}{27}\left[1+\ln 27+\frac{(\ln 27)^2}{2!}\right].$

3. A random variable uniformly distributed on $[a,b]$ has mean $\dfrac{a+b}{2}$ and variance $\dfrac{(b-a)^2}{12}$. Hence, we have that $b-a=6$, and $b+a=2$, giving $a=-2, b=4$. The pdf is thus as shown below.



(**a**) By inspection, $P\{\mathcal{X}<0\} = \frac{1}{3}$.

(**b**) $\mathsf{E}[|\mathcal{X}|] = \dfrac{1}{6}\left[\displaystyle\int_{-2}^{0} -u\,du + \int_{0}^{4} u\,du\right] = \dfrac{1}{6}\left[\dfrac{-u^2}{2}\Big|_{-2}^{0} + \dfrac{u^2}{2}\Big|_{0}^{4}\right] = \dfrac{1}{6}[2+8] = \dfrac{5}{3}.$

(**c**) $\mathcal{Y} = |\mathcal{X}|$ takes on values in $[0,4]$, and hence $F_{\mathcal{Y}}(v) = 0$ for $v<0$ and $F_{\mathcal{Y}}(v)=1$ for $v>4$.
For any $v, 0\leq v\leq 2, F_{\mathcal{Y}}(v) = P\{\mathcal{Y}\leq v\} = P\{[-v\leq \mathcal{X}\leq v\} = v/3$.
For any $v, 2\leq v\leq 4, F_{\mathcal{Y}}(v) = P\{\mathcal{Y}\leq v\} = P\{\mathcal{X}\leq v\} = (v+2)/6$.
Hence, $f_{\mathcal{Y}}(v) = \dfrac{d}{du}F_{\mathcal{Y}}(v) = \begin{cases} \frac{1}{3}, & 0\leq v\leq 2,\\ \frac{1}{6}, & 2<v\leq 4,\\ 0, & \text{elsewhere.}\end{cases}$
It is easy to verify that this is a valid pdf, and thus we have not made any obvious errors.

4. $P\{|\mathcal{X}-4|>3\} = 1-P\{|\mathcal{X}-4|\leq 3\} = 1-P\{1\leq\mathcal{X}\leq 7\} = 1-\left[\Phi\left(\dfrac{7-2}{5}\right)-\Phi\left(\dfrac{1-2}{5}\right)\right]$
$= 1-[\Phi(1)-\Phi(-0.2)] = 2-\Phi(1)-\Phi(0.2) = 2-0.8413-0.5793 = 0.5794.$

$$P\{\mathcal{X} < 3 | \mathcal{X} > 2\} = \frac{P(\{\mathcal{X} < 3\} \cap \{\mathcal{X} > 2\})}{P\{\mathcal{X} > 2\}} = \frac{P\{2 < \mathcal{X} < 3\}}{P\{\mathcal{X} > 2\}} = \frac{\Phi((3-2)/5) - \Phi((2-2)/5)}{\Phi((2-2)/5)}$$
$$= \frac{\Phi(0.2) - \Phi(0)}{\Phi(0)} = 2 \cdot [\Phi(0.2) - 0.5] = 2 \times 0.5793 - 1 = 0.1586.$$