Assigned: Wednesday, April 30
Due: Wednesday, May 7
Reading: Yates and Goodman: Chapters 5, 7 and 9.4–9.6
Problems:

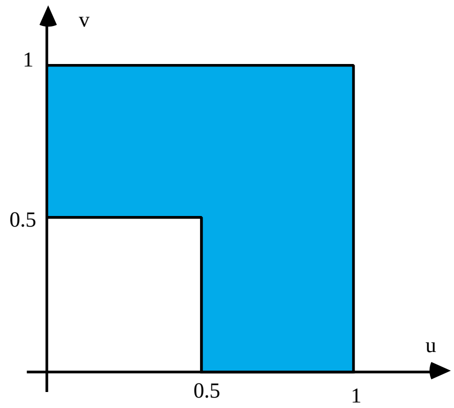
1. Let $\Phi_{\mathbf{X}}(\omega)$ denote the characteristic function of a random variable **X**.

(a) What is the value of $\Phi_{\mathbf{X}}(0)$? Does it matter whether **X** is discrete or continuous?

(b) Assume that $\Phi_{\mathbf{X}}(\omega)$ is differentiable at $\omega = 0$. What is the value of $\frac{d}{d\omega}\Phi_{\mathbf{X}}(\omega)$ at $\omega = 0$?

(c) What is the characteristic function of a Cauchy random variable? [Hint: the answer can be found in the inside front cover of the Kudeki-Munson Lecture Notes for ECE 210] Why can't the method of part (b) be used to compute the mean of a Cauchy random variable?

2. The random point $(\mathbf{X}, \mathbf{Y})$ is uniformly distributed on the shaded region shown below.
(a) Find the marginal pdf $f_{\mathbf{X}}(u)$ of the random variable **X**.
(b) *Write down* the marginal pdf $f_{\mathbf{Y}}(v)$ of the random variable **Y** from your answer to part (b).
(c) Find $P\{\mathbf{X} < \mathbf{Y} < 2\mathbf{X}\}$.
(d) What is $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)$, the conditional pdf of **X** given that $\mathbf{Y} = \alpha$, if α satisfies $0 < \alpha < 1/2$?
What is $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)$, the conditional pdf of **X** given that $\mathbf{Y} = \alpha$, if α satisfies $1/2 < \alpha < 1$?
Now, apply the theorem of total probability to compute the unconditional pdf of **X** from $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)$. Do you get the same answer as in part (a)?



3. Let $E[\mathbf{X}] = 1$, $E[\mathbf{Y}] = 4$, $\text{var}(\mathbf{X}) = 4$, $\text{var}(\mathbf{Y}) = 9$, and $\rho_{\mathbf{X},\mathbf{Y}} = 0.1$
(a) If $\mathbf{Z} = 2(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})$, what is $E[\mathbf{Z}]$?
(b) If $\mathbf{T} = 2\mathbf{X}+\mathbf{Y}$ and $\mathbf{U} = 2\mathbf{X}-\mathbf{Y}$, what is $\text{cov}(\mathbf{T}, \mathbf{U})$?
(c) If $\mathbf{W} = 3\mathbf{X} + \mathbf{Y} + 2$, find $E[\mathbf{W}]$ and $\text{var}(\mathbf{W})$.
(d) If **X** and **Y** are jointly Gaussian random variables, and **W** is as defined in (c), what is $P\{\mathbf{W} > 0\}$?

4. This problem has three independent parts. Do not apply the numbers from one part to the others.
(a) If $\text{var}(\mathbf{X} + \mathbf{Y}) = 36$ and $\text{var}(\mathbf{X} - \mathbf{Y}) = 64$, what is $\text{cov}(\mathbf{X}, \mathbf{Y})$? If you are also told that $\text{var}(\mathbf{X}) = 3 \bullet \text{var}(\mathbf{Y})$, what is $\rho_{\mathbf{X},\mathbf{Y}}$?
(b) If $\text{var}(\mathbf{X} + \mathbf{Y}) = \text{var}(\mathbf{X} - \mathbf{Y})$, are **X** and **Y** uncorrelated ?
(c) If $\text{var}(\mathbf{X}) = \text{var}(\mathbf{Y})$, are **X** and **Y** uncorrelated ?

5. Consider the random point $(\mathbf{X}, \mathbf{Y})$ of Problem 2 above.
(a) Compute $E[\mathbf{X}]$ and $\text{var}(\mathbf{X})$.
(b) Explain why the random variable **Y** has the same mean and variance as **X**.
(c) Compute $E[\mathbf{XY}]$ and hence find $\text{cov}(\mathbf{X}, \mathbf{Y})$.
should hold. ***Is*** the above equation satisfied by the numerical values you obtained?

(d) The conditional pdf of **X** given **Y** = α was obtained in Problem 2 above, and it is easy to see that the conditional pdf of **Y** given **X** = α is similar. Now, the **best** (least mean-square error) estimate of **Y** given **X** = α is the mean of the conditional pdf of **Y** given **X** = α. Thus, if **X** has value $\alpha \le 0.5$, then $\hat{\mathbf{Y}}$, the best estimate of **Y**, is 0.75 while if **X** has value $\alpha > 0.5$, then $\hat{\mathbf{Y}} = 0.5$. Now, the **best linear** (least mean-square error) estimate of **Y** (given that **X** is known to have value α) is $\tilde{\mathbf{Y}} = a + b\alpha$ where a and b are given in Theorem 9.11 of Y&G. Compute a and b, and draw a graph showing the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ as functions of α. (Remember that $0 \le \alpha \le 1$). For what value(s) of α are the two estimates the same?

(e) Since the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ depend on the value of **X**, they really are *functions* of **X**, that is, they are *random variables* that can be expressed as $\hat{\mathbf{Y}} = \begin{cases} 0.75, & 0 \le \mathbf{X} \le 0.5, \\ 0.5, & 0.5 < \mathbf{X} \le 1 \end{cases}$ and $\tilde{\mathbf{Y}} = a + b\mathbf{X}$. What are the average and the mean-square errors of each estimate? That is, what are the values of $E[(\mathbf{Y} - \hat{\mathbf{Y}})]$, $E[(\mathbf{Y} - \tilde{\mathbf{Y}})]$, $E[(\mathbf{Y} - \hat{\mathbf{Y}})^2]$, and $E[(\mathbf{Y} - \tilde{\mathbf{Y}})^2]$?