Assigned: Wednesday, April 23
Due: Wednesday, April 30
Reading: Yates and Goodman: Chapters 5 and 7
Problems:

1. Let (**X**, **Y**) have joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v)$ that is a circularly symmetric function, i.e., $f_{\mathbf{X},\mathbf{Y}}(u, v)$ can be expressed as g(r) where $r = \sqrt{u^2+v^2}$. The random point (**X**, **Y**) is at distance $\mathbf{R} = \sqrt{\mathbf{X}^2+\mathbf{Y}^2}$ from the origin.

(a) Show that for $\alpha \geq 0$, $P\{\mathbf{R} \leq \alpha\} = \int_0^\alpha 2\pi \cdot r \cdot g(r)\, dr$.

(b) Use the formula for differentiating an integral that we studied in class to show that $f_{\mathbf{R}}(\alpha) = \frac{d}{du} P\{\mathbf{R} \leq \alpha\} = 2\pi \cdot \alpha \cdot g(\alpha)$ for $\alpha > 0$.

(c) Use the result of part (b) to deduce the pdf of **R** if (**X**,**Y**) is uniformly distributed on the unit disc, viz. the interior of the circle of radius 1 centered at the origin

(d) Now suppose that (**X**, **Y**) has joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} C\sqrt{1-u^2-v^2}, & u^2+v^2 < 1, \\ 0, & \text{elsewhere.} \end{cases}$

What is the value of C?

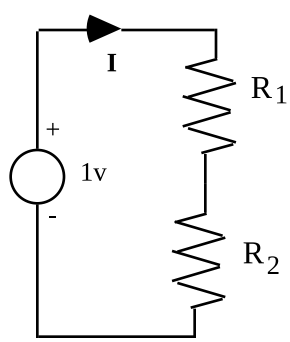
(e) Find $P\{\mathbf{X}^2+\mathbf{Y}^2 < 0.25\}$.

2. One way of defining a "random chord" of a circle is to choose the midpoint of the chord to be anywhere inside the circle with equal probability. The chord is perpendicular to the diameter of the circle that passes through the chosen point. Thus, let the random point (**X**,**Y**) — denoting the midpoint of the chord — be uniformly distributed on the unit disc of Problem 1(c).

(a) Find the probability that the length **L** of the random chord is greater than the side of the equilateral triangle inscribed in the circle. [Hint: draw the triangle and the circle!]

(b) Express **L** as a function of the random variable (**X**, **Y**) and find the probability density function for **L**.

(c) Find the average length of the chord, i.e. find E[**L**].



3. Two resistors are connected in series to a one-volt voltage source as shown in the right-hand diagram above. Suppose that the resistance values $\mathbf{R}_1$ and $\mathbf{R}_2$ (measured in ohms) are independent random variables, each uniformly distributed on the interval (0, 1). Find the pdf $f_{\mathbf{I}}(a)$ of the current **I** (measured in amperes) in the circuit.

4. Let (**X**, **Y**) have joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} 2u, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$

Find the pdf of $\mathbf{Z} = \mathbf{X}^2\mathbf{Y}$.

5. This problem is based on two results, one proved in class, viz. if $\mathbf{X}$ is $N(0,\sigma^2)$, then $\mathbf{X}^2$ has gamma pdf with parameter $(1/2, 1/2\sigma^2)$ and another not explicitly stated in Y&G viz. if $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are independent gamma random variables with parameters (t_1, λ), (t_2, λ), $\ldots$, (t_n, λ) respectively, then $\Sigma\, \mathbf{X}_i$ is a gamma random variable with parameters $(\Sigma\, t_i, \lambda)$

(a) Now, suppose that $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are independent $N(0,\sigma^2)$ random variables. What are the pdfs of $\mathbf{X}^2$, $\mathbf{Y}^2$, and $\mathbf{Z}^2$? Are these random variables independent also?

(b) ***State*** what the *type* of pdf of $\mathbf{W} = \mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2$ is, and write down *explicitly* the exact pdf of $\mathbf{W}$.

(c) Prove that $E[\mathbf{W}] = 3\sigma^2$. If you actually evaluated an integral to get this answer instead of using LOTUS, shame on you!

(d) In a physical application, $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ represent the velocity (measured along three perpendicular axes) of a gas molecule of mass m. Thus, $\mathbf{H} = (1/2)m\mathbf{W}$ is the kinetic energy of the particle, and an important axiom of statistical mechanics asserts that the average kinetic energy is $E[\mathbf{H}] = E[(1/2)m\mathbf{W}] = (1/2)mE[\mathbf{W}] = (3/2)m\sigma^2 = (3/2)kT$ where k is Boltzmann's constant and T is the absolute temperature of the gas in °K. (Note that the average energy is (1/2)kT per dimension.) Show that the kinetic energy $\mathbf{H}$ has the Maxwell-Boltzmann pdf $f_{\mathbf{H}}(\beta) = \frac{2}{\sqrt{\pi}}(kT)^{-3/2}\sqrt{\beta}\cdot\exp(-\beta/kT)$, $\beta > 0$.

(e) $\mathbf{V} = \sqrt{\mathbf{W}} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2}$ is the "speed" of the molecule. Show that the pdf of $\mathbf{V}$ is

$$f_{\mathbf{V}}(\gamma) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2}\gamma^2\exp\left(-\frac{m\gamma^2}{2kT}\right),\ \gamma > 0.$$

(f) What is the average speed of the molecule?

6. The number of hours $\mathbf{R}$ that a student spends **r**eading about probability in preparation for the ECE 313 Final Examination and the number of hours $\mathbf{S}$ that the student spends **s**leeping can be modeled as random variables with joint probability density function

$$f_{\mathbf{R},\mathbf{S}}(x,y) = \begin{cases} K, & 10 \le x + y \le 20,\ x \ge 0,\ y \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of K?

(b) What is the marginal pdf of $\mathbf{R}$?

(c) Unfortunately, the more the student tries to read about probability, the more confused the student gets. Also, the less the student sleeps, the more tired the student gets. As a result, the student's *percentage* score $\mathbf{T}$ on the Final Exam is related to $\mathbf{S}$ and $\mathbf{R}$ via the equation

$$\mathbf{T} = 50 + 2.5(\mathbf{S} - \mathbf{R}).$$

Find the pdf of $\mathbf{T}$.

(d) **Noncredit exercise:** Should $\mathbf{S}$ have denoted **s**tudying and $\mathbf{R}$ denoted **r**esting instead?