Assigned: Wednesday, April 16
Due: Wednesday, April 23
Reading: Yates and Goodman: Chapters 5 and 7
Problems:

1. (postponed from Problem Set #11)
The discrete random variables **X** and **Y** have joint pmf $p_{\mathbf{X},\mathbf{Y}}(u,v)$ given by

4	0	1/12	1/6	1/12
3	1/6	1/12	0	1/12
–1	1/12	1/6	1/12	0
v↑ / u→	0	1	3	5

(a) Find the marginal pmfs $p_{\mathbf{X}}(u)$ and $p_{\mathbf{Y}}(v)$ of **X** and **Y**.
(b) Are the random variables **X** and **Y** independent ?
(c) Find $P\{\mathbf{X} \le \mathbf{Y}\}$ and $P\{\mathbf{X} + \mathbf{Y} \le 8\}$.
(d) Find $p_{\mathbf{X}|\mathbf{Y}}(u|3)$, $E[\mathbf{X}|\mathbf{Y}=3]$, and $\mathrm{var}(\mathbf{X}|\mathbf{Y}=3)$.

2. The joint pdf of **X** and **Y** is given by

$$f_{\mathbf{X},\mathbf{Y}}(u,v) = \begin{cases} c\bullet(v^2 - u^2)\bullet\exp(-v), & -v \le u \le v,\ 0 < v < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is the value of c?
(b) Find $f_{\mathbf{X}}(u)$ and $f_{\mathbf{Y}}(v)$, the marginal pdfs of **X** and **Y** respectively.
(c) Find $E[\mathbf{X}]$.

3. The jointly continuous random variables **X** and **Y** have joint pdf given by

$$f_{\mathbf{X},\mathbf{Y}}(u,v) = \begin{cases} 2\exp-(u+v), & 0 < u < v < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Sketch the u-v plane and indicate on it the region over which $f_{\mathbf{X},\mathbf{Y}}(u,v)$ is nonzero.
(b) Find the marginal pdfs of **X** and **Y**.
(c) Are the random variables **X** and **Y** independent ?
(d) Find $P\{\mathbf{Y} > 3\mathbf{X}\}$.
(e) For $\alpha > 0$, find $P\{\mathbf{X} + \mathbf{Y} \le \alpha\}$.
(f) Use the result in part (e) to determine the pdf of the random variable $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$.

4. The jointly continuous random variables **X** and **Y** have joint pdf

$$f_{\mathbf{X},\mathbf{Y}}(u,v) = \begin{cases} 1/2, & 0 \le u < 1,\ 0 \le v < 1, \text{ and } 0 \le u+v < 1 \\ 3/2, & 0 \le u < 1,\ 0 \le v < 1, \text{ and } 1 \le u+v < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{\mathbf{X}}(u)$, $P\{\mathbf{X} + \mathbf{Y} \le 3/2\}$ and $P\{\mathbf{X}^2 + \mathbf{Y}^2 \ge 1\}$.

5. Let **X** and **Y** denote *independent* $N(0, \sigma^2)$ variables.
(a) What is the joint pdf $f_{\mathbf{X},\mathbf{Y}}(u,v)$ of **X** and **Y**?
(b) Sketch the u-v plane and indicate on it the region over which you need to integrate the joint pdf in order to find $P\{\mathbf{X}^2 + \mathbf{Y}^2 > \alpha^2\}$. Then, compute $P\{\mathbf{X}^2 + \mathbf{Y}^2 > \alpha^2\}$. Hint: read the Solutions to Problems 4(b) of Problem Set #1 and Problem 2(a) of Problem Set #10.
(c) Now, let $\mathbf{Z} = \mathbf{X}^2 + \mathbf{Y}^2$ denote the squared distance of the random point (**X**, **Y**) from the origin. Use the result of part (b) to deduce the pdf of **Z**.
From here onwards, assume $\sigma^2 = 1$ so that **X** and **Y** are independent *unit* Gaussian RVs.

(d) Express P{|**X**| > α} in terms of the complementary unit Gaussian CDF function Q(x), and use this to write P{|**X**| > α, |**Y**| > α} in terms of Q(x). (Remember commas mean intersections).

(e) Sketch the u-v plane and show on it the region over which you must integrate the joint pdf to find P{|**X**| > α, |**Y**| > α}. Compare the sketches in parts (b) and (d) to deduce that P{|**X**| > α, |**Y**| > α} ≤ P{$\mathbf{X}^2 + \mathbf{Y}^2 > 2\alpha^2$}.

(f) Show that the inequality of part (d) implies that Q(x) ≤ (1/2)•exp($-x^2/2$) as was proved earlier in Problem 2(b) of Problem Set #10.