Assigned: Wednesday, April 9
Due: Wednesday, April 16
Reading: Yates and Goodman: Chapter 5

Reminder: **Hour Exam II** is scheduled for **Monday April 14, in class.**
One 8.5" by 11" sheet of notes is allowed.
Calculators, laptop computers, Palm Pilots, pagers, SMS, etc are not allowed.
The material covered on Problem Sets 5-10 is included on the exam. It will also help you if you make an early start on *this* Problem Set since it will provide you with valuable practice in problem-solving.
Lecture material covered on the Exam is Lectures 12-14 and 18-28 of the Powerpoint slides available on the class web page. Also, discrete random variables treated earlier in the course re-appeared in Poisson random processes, DeMoivre-LaPlace theorem etc., so don't neglect them entirely.
You should have the pdfs/pmfs, means, variances etc. of binomial, Poisson, geometric, negative binomial; uniform, exponential, gamma, and Gaussian random variables on your sheet of notes

Problems:

1. ["A chip off the old block"] We return to Problem 4 of Problem Set #9 where you were asked to show that **Y**, the lifetime of a TMR system (with infallible majority gate and VLSI chips with lifetimes that are exponential random variables with parameter $\lambda = -\ln 0.999$/week), has *complementary* CDF $1 - F_{\mathbf{Y}}(t) = P\{\mathbf{Y} > t\} = 3\exp(-2\lambda t) - 2\exp(-3\lambda t),\ t > 0$.

(a) Sketch $f_{\mathbf{Y}}(t)$, the *pdf* of **Y** and also $f_{\mathbf{X}}(t)$, the pdf of **X**, the lifetime of a chip, on the same set of axes. Use Mathematica/Matlab etc. for this. Don't just wing it!

(b) Find all values of t for which $f_{\mathbf{Y}}(t) = f_{\mathbf{X}}(t)$. An analytical answer is desired. Don't just guesstimate the locations of the crossing-points of the two pdf curves from your graph in part (a). [Hint: set $x = \exp(-\lambda t)$]. For small values of t, which is larger: $f_{\mathbf{Y}}(t)$ or $f_{\mathbf{X}}(t)$?

(c) Sketch the hazard rate $h_{\mathbf{Y}}(t)$ of the TMR system and the hazard rate $h_{\mathbf{X}}(t)$ of a VLSI chip on the same set of axes. For what values of t does the TMR system have a larger hazard rate than the VLSI chip?

(d) Which is more likely to fail in the next minute: a one-week old TMR system or a one-week old VLSI chip?

(e) Which is more likely to fail in the next minute: a twenty-year old TMR system or a twenty-year old VLSI chip? For the purposes of this problem, twenty years = 1040 weeks, (which reminds me: income tax returns must be filed by April 15…)

2. The random variable **X** has probability density function $f_{\mathbf{X}}(u) = \begin{cases} 2(1-u), & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

Let $\mathbf{Y} = (1 - \mathbf{X})^2$.

(a) What is the CDF $F_{\mathbf{Y}}(v)$ of the random variable **Y**? Be sure to specify the value of $F_{\mathbf{Y}}(v)$ for all v, $-\infty < v < \infty$.

(b) What is the pdf $f_{\mathbf{Y}}(v)$ of the random variable **Y**? Be sure to specify the value of $f_{\mathbf{Y}}(v)$ for all v, $-\infty < v < \infty$.

3. ["Give me an A! Give me a D! Give me a converter! What have we got? An A/D converter! Go Team!"] A signal **X** is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value **Y** (where $\mathbf{Y} = \alpha$ if $\mathbf{X} > 0$ and $\mathbf{Y} = -\alpha$ if $\mathbf{X} \le 0$) is used. Note that **Y** is a *discrete* random variable.

(a) What is the pmf of **Y**?

(b) Suppose that $\alpha = 1$. If the signal **X** happens to have value 1.29, what is the error made in representing **X** by **Y**? What is the squared-error? Repeat for the case when **X** happens to have value $\pi/4$ and when **X** happens to have value $-\pi/4$.

(c) We wish to design the quantizer so as to minimize the squared-error. However, since **X** (and **Y**) are random, we can only minimize the squared-error in the probabilistic (that is, average) sense. Now, part (b) shows that the squared-error depends on the value of **X**, and can be expressed as $\mathbf{Z} = (\mathbf{X} - \mathbf{Y})^2 = g(\mathbf{X}) = \begin{cases} (\mathbf{X} - \alpha)^2 & \text{if } \mathbf{X} > 0 \\ (\mathbf{X} + \alpha)^2 & \text{if } \mathbf{X} \le 0. \end{cases}$

So we want to choose α so that E[**Z**] is as small as possible. Use LOTUS to e-zily find E[**Z**] as a function of α, and then find the value of α that minimizes E[**Z**]. [Hint: read the solutions to Problem 2(a) of Problem Set #10.]

(d) We now get more ambitious and use a 3-bit A/D converter which first quantizes **X** to the nearest integer **W** in the range –3 to +3. Thus, **W** = 3 if **X** ≥ 2.5, **W** = 2 if 1.5 ≤ **X** < 2.5, etc. Note that **W** is a discrete random variable. Find the pmf of **W**.

(e) The output of the A/D converter is a 3-bit 2's complement representation of **W**. Suppose that the output is ($\mathbf{Z}_2$, $\mathbf{Z}_1$, $\mathbf{Z}_0$). What is the pmf of $\mathbf{Z}_2$? of $\mathbf{Z}_1$? of $\mathbf{Z}_0$?

(f) **Noncredit exercise (but a real-life engineering problem!):** Suppose that **W** takes on values $-3\alpha, -2\alpha, -\alpha, 0, +\alpha, +2\alpha, +3\alpha$ and quantization is as before: **X** is mapped to the nearest **W** value. What value of α minimizes $E[(\mathbf{X} - \mathbf{W})^2]$?

4. The discrete random variables **X** and **Y** have joint pmf $p_{\mathbf{X},\mathbf{Y}}(u,v)$ given by

4	0	1/12	1/6	1/12
3	1/6	1/12	0	1/12
–1	1/12	1/6	1/12	0
v↑ / u→	0	1	3	5

(a) Find the marginal pmfs $p_{\mathbf{X}}(u)$ and $p_{\mathbf{Y}}(v)$ of **X** and **Y**.

(b) Are the random variables **X** and **Y** independent ?

(c) Find P{**X** ≤ **Y**} and P{**X** + **Y** ≤ 8}.

(d) Find $p_{\mathbf{X}|\mathbf{Y}}(u|3)$, E[**X**|**Y**=3], and var(**X**|**Y**=3).