Assigned: Wednesday, April 2
Due: Wednesday, April 9
Reading: Yates and Goodman: Chapter 4 and 6.5
Problems:

1. Problem 4.5.4 on page 162 of Yates and Goodman, viz. The peak temperature **T** in °F on a July day in Antarctica is a Gaussian random variable with variance 225. With probability 1/2, the temperature exceeds 10°F. What is P{**T** > 32°F}, the probability that the temperature is above freezing? What is P{**T** < 0°F}? What is P{**T** > 60°F}?

2. Let **X** denote a unit Gaussian random variable with pdf $\varphi(u)$ and CDF $\Phi(u)$.

(a) What is the derivative of $\exp(-u^2/2)$ with respect to u? Use this result to find E[|**X**|].

Now, let $Q(x) = \int_x^\infty \varphi(u)\,du = \int_x^\infty (\sqrt{2\pi})^{-1}\exp\left(-\frac{u^2}{2}\right)du = 1 - \Phi(x)$.

(b) A useful bound when x is small is $Q(x) \le (1/2)\exp(-x^2/2)$ for $x \ge 0$ in which equality holds only at x = 0. Derive this bound by first showing that $t^2 - x^2 > (t-x)^2$ for $t > x > 0$ and then applying this result to $\exp(x^2/2)Q(x) = \int_x^\infty (\sqrt{2\pi})^{-1}\exp\left[-\left(\frac{t^2 - x^2}{2}\right)\right]dt$

3. Do **either** part (a) **or** part (b). Then do parts (c)–(e).

(a) Attach to your homework a **photocopy** of your calculator's manual page(s) that explains which **formula** your calculator uses to compute Q(x). Reading the page might help too! Note: I **do not want** to know **which buttons** you have to press in order to find Q(x); I want to know **what formula** your calculator uses internally to find Q(x).
The xerographically–challenged are permitted to just copy the relevant formulas to their homework. **NEXT:** press the appropriate buttons to **find Q(5).**
If your calculator cannot compute Q(x), or if the manual does not state what formula is used to calculate Q(x) but just tells you which buttons to press, or if you have lost the manual, do part (b) instead.

(b) Read Chapter 26.2 of Abramowitz and Stegun (*reference* book *(not a **reserve** book)* in Grainger Engineering Library), and use Equation 26.2.17 to **calculate Q(5).**
This formula is also given on Slide 30 of Lecture 26 in the Powerpoint slides.

(c) The number found in part (a) or (b) is just an *approximation* to the value of Q(5). Use the maximum error specification to find the *range* in which the actual value of Q(5) must necessarily lie. What is the *maximum relative error* in the approximation to Q(5) that you found in part (a) or (b)? Note: the relative error is defined as $\frac{|\text{true value} - \text{computed value}|}{\text{true value}}$ expressed as a percentage.

(d) On p. 972, Abramowitz and Stegun give the value of $-\log_{10}Q(5)$ to be 6.54265…. Blindly trust your calculator to do the exponentiation correctly and find the *actual relative error* in the approximation to Q(5) that you found in part (a) or (b). What would the actual relative error have been if you had simply used the upper bound $\varphi(5)/5$ as an approximation to Q(5)? What if you had used the lower bound $(1/5 - 1/5^3)\varphi(5)$ as an approximation to Q(5)?

(e) Explain why the "much easier" Equation 26.2.18 of Abramowitz and Stegun is not particularly useful for computing Q(5).

4. A signal $x(t) = \exp(-\pi t^2)$, $-\infty < t < \infty$, is passed through a low-pass filter whose transfer function is $H(f) = \begin{cases} 2, & -1 \le f \le 1, \\ 0, & 1 < |f| < \infty. \end{cases}$ Let y(t) denote the output of the filter. Compute the value of y(0). A numerical answer is desired. [Hint: $X(f) = \exp(-\pi f^2)$, $-\infty < f < \infty$.]

5. $\mathbf{X}$ is a continuous random variable with pdf $f_{\mathbf{X}}(u) = 0.5 \exp(-|u|)$, $-\infty < u < \infty$.

(a) What is the value of $P\{\mathbf{X} \le \ln 2\}$?

(b) Find the conditional probability that $P\{|\mathbf{X}| \le \ln 2\}$ given that $\{\mathbf{X} \le \ln 2\}$.

(c) Find the numerical value of $P\{\cos(\pi\mathbf{X}/2) < 0\}$.

(d) Now suppose that $\mathbf{X}$ denotes the voltage applied to a semiconductor diode, and that the current $\mathbf{Y}$ is given by $\mathbf{Y} = e^{\mathbf{X}} - 1$. Find the pdf of $\mathbf{Y}$.

6. The radius of a sphere is a random variable $\mathbf{R}$ with pdf $f_{\mathbf{R}}(\rho) = \begin{cases} 3\rho^2, & 0 < \rho < 1, \\ 0 & \text{elsewhere.} \end{cases}$

(a) Use LOTUS to find the average radius, average volume and average surface area of the sphere. Does a sphere of average radius also have average volume? Does a sphere of average radius also have average surface area?

(b) Find the CDF $F_{\mathbf{V}}(\alpha)$ and pdf $f_{\mathbf{V}}(\alpha)$ of $\mathbf{V}$, the volume of the sphere.

(c) Find $E[\mathbf{V}]$ directly from this pdf. Do you get the same answer as in part (a)? Why not?

(d) If the sphere is made of metal and carries an electrical charge of Q coulombs, what is the CDF $F_{\mathbf{S}}(x)$ and the pdf $f_{\mathbf{S}}(x)$ of the surface charge density $\mathbf{S}$ on the sphere ?