Assigned: Wednesday, March 19
Due: Wednesday, April 2
Reading: Yates and Goodman: Chapter 4
Reminder: Spring Break will be much more enjoyable if you go easy on the beer and heavy on the suntan lotion instead of the other way around…

Problems:

1. ["Extra! Extra! Read all about it!"] A newsboy purchases H newspapers for c_2 cents each and sells them for c_3 cents each. He can return unsold papers to the publisher for c_1 cents each. Note that $c_1 < c_2 < c_3$. The daily demand **X** for papers is a (nonnegative) integer-valued random variable with pmf $p_X(u)$ and CDF $F_X(u)$.

(a) What is the probability that he sells all H newspapers? Express your answer in terms of $p_X(u)$ and also in terms of $F_X(u)$.

(b) Let **Z** denote the daily profit (in cents) that the newsboy makes. Write an expression for **Z** in terms of **X** and H

(c) Use LOTUS to write an expression for his average daily profit. Your answer will depend on H, so call the expression for the **average** daily profit the function g(H).

(d) The newsboy has been buying H papers for some months and making an average profit g(H) each day. One day, he decides to buy one extra paper. What is the probability that he can sell this extra paper? Show that he makes an average **additional** profit of $(c_3 - c_2) - (c_3 - c_1)F_X(H)$ from the extra paper. Call this A(H).

(e) Show that the average **additional** profit $A(H) = (c_3 - c_2) - (c_3 - c_1)F_X(H)$ satisfies

$$\ldots A(H-1) \geq A(H) \geq A(H+1) \ldots,$$

that is, on average, each extra newspaper brings in smaller extra profit than the previous one. [Hint: F_X is a non-decreasing function…] This is called the law of diminishing returns.

(f) Show that for sufficiently large H, A(H) is negative so that the newsboy loses money (on the average) by buying too many extra papers.

(g) How many papers should he purchase to maximize his average profit?

2. **X** is uniformly distributed on [–1, +1].

(a) If $\mathbf{Y} = \mathbf{X}^2$, what are the mean and variance of **Y**?

(b) If $\mathbf{Z} = g(\mathbf{X})$ where $g(u) = \begin{cases} u^2, & u \geq 0, \\ -u^2, & u < 0, \end{cases}$ use LOTUS (or the EZ method) to find E[**Z**]

(c) On a completely unrelated LOTUSian question, if **X** is a geometric random variable with parameter 1/2, and $\mathbf{Y} = \sin(\pi \mathbf{X}/2)$, what is the value of E[**Y**]?

3. We return to Problem 6 of Problem Set #8. Suppose that the owner makes a gross profit of \$0.64 for each gallon of gasoline sold. Let **Y** denote the amount of gasoline *sold* per week.

(a) How is **Y** related to **X**, the weekly *demand* for gasoline? (Hint: the owner cannot sell more gasoline each week than the tank can hold!)

(b) What is the **average** weekly gross profit?

(c) Suppose that the owner pays \$20C as weekly rent on a tank of capacity 1000C gallons. Note that $0 \leq C \leq 1$. (Why is a tank larger than 1000 gallons not needed?) What is the **average** weekly **net** profit and what value of C maximizes the average weekly net profit?

4. The lifetime of a VLSI chip can be modeled as an exponential random variable with parameter $\lambda = -\ln 0.999$/week.

(a) What is the average lifetime (in weeks)? What is the median lifetime?

(b) What is the probability that the chip lasts for at least one week?

Now suppose that three identical chips are organized into a triple–modular–redundancy (TMR) system in which we assume that the majority–logic gate cannot fail. Furthermore, we assume that the three chips fail independently of one another, that is, if their lifetimes are $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$, then the events $\{\mathbf{X}_1 > t_1\}$, $\{\mathbf{X}_2 > t_2\}$, and $\{\mathbf{X}_3 > t_3\}$ are independent for all t_1, t_2, t_3. Let **Y** denote the length of time for which the TMR system functions correctly.

(c) Express the event $\{\mathbf{Y} > t\}$ occurred in terms of unions, intesections and complements of the events $\{\mathbf{X}_1 > t\}$, $\{\mathbf{X}_2 > t\}$, and $\{\mathbf{X}_3 > t\}$.

(d) Show that $P\{\mathbf{Y} > t\} = 3\exp(-2\lambda t) - 2\exp(-3\lambda t)$ and use this result to find the average lifetime and the median lifetime of the TMR system. [Hint: $E[\mathbf{Y}]$ is the integral of $P\{\mathbf{Y} > t\}$ over the positive real line!] . Compare your answers to those in part (a). Do the results surprise you? Is the TMR system improving performance the way it is alleged to?

(e) What is the probability that the TMR system functions correctly for at least one week? Compare this answer to that of part (b). Do you think that the TMR system is more reliable or less reliable?

(f) Find t such that $P\{\mathbf{Y} > t\} = 0.999$ and compare the answer to that of part (b). Has the TMR system improved performance?

5. Consider a Poisson process with arrival rate μ. Let **X** denote the time of the first arrival after $t = 0$, and let τ denote a nonnegative real number.

(a) What is $P\{\mathbf{X} > \tau\}$?

(b) What is the probability density function of **X**?

(c) Let A denote the event that there are exactly four arrivals in the time interval (0, 6]. What is P(A)?

(d) Let B denote the event that there are exactly two arrivals in the interval (4, 10]. What is P(AB)? [Hint: Read the Solutions to Problem 2(d) on Hour Exam I first.]

(e) What is the conditional probability that $\{\mathbf{X} > \tau\}$ *given* that the event A occurred? Be sure to give the answer for all nonnegative values of τ.