Assigned: Wednesday, February 26
Due: Wednesday, March 5
Reading: Yates and Goodman: Chapters 1, 2 and 3.6
Noncredit Exercises: (Do not turn these in) p. 20: Quiz 1.5; p. 27: Quiz 1.7; p. 79: Quiz 2.9; pp. 38-39: Problems 1.5.1 – 1.5.6, p. 86: 2.9.1 – 2.9.8

REMINDER: HOUR EXAM I is on Monday March 3 (in class). One 8.5" × 11" sheet of notes is permitted (both sides can be used). For details of coverage, see the class web page under "Late Breaking News"

Problems:

1. [“I am from Iowa; I only work in outer space…”] Each box of Cornies, the breakfast of silver medalists, contains one picture, which is of Luke Skywalker with probability 2/3 and of Darth Vader with probability 1/3, independently of which picture is in any other box of Cornies. Little Jimmy Kirk of Cedar Rapids, Iowa, asks his mother to buy boxes of Cornies until he has at least one picture of both beings, and his mother agrees to do so.

(a) What is the minimum number of boxes of Cornies that Mrs Kirk must buy?

(b) Let **X** denote the number of boxes of Cornies Mrs Kirk purchases until such time as Jimmy has acquired at least one picture of each of the two entities. What is the pmf of **X**? Verify that the total probability mass specified by your pmf does equal 1.

(c) What is the conditional pmf of **X** given that the first box contained a picture of Luke? What is the conditional pmf of **X** given that the first box contained a picture of Darth?

(d) Use the theorem of total probability to compute the unconditional pmf of **X** from the conditional pmfs found in part (c). Do you get the same answer as in part (b)? Why not?

(e) Find the mean and variance of **X**. [Hint: it might be easier to determine the conditional means and variances and then combine them to obtain the unconditional mean and variance]

2. (Remember: $99\frac{44}{100}$% of all statistics are made up by the writer) The experiment consists of picking a flight at random from all the United Airlines and America West flights landing at Chicago, Los Angeles, Phoenix, San Diego, or San Francisco. Let U and W respectively denote the event that the chosen flight is an United Airlines or an America West flight, let C, L, X, D, and F respectively denote the event that the chosen flight is landing at Chicago, Los Angeles, Phoenix, San Diego, or San Francisco, and let T denote the event that the chosen flight is on time. The conditional probabilities of on-time arrival are as follows:

$P(T|UC) = 0.85$, $P(T|UL) = 0.92$, $P(T|UX) = 0.95$, $P(T|UD) = 0.91$, $P(T|UF) = 0.83$,
$P(T|WC) = 0.78$, $P(T|WL) = 0.88$, $P(T|WX) = 0.92$, $P(T|WD) = 0.85$ $P(T|WF) = 0.73$.

(a) Based on this data, which airline would you say has better on-time performance? Does the answer depend on which airport you are talking about?

(b) Use the fact that {C, L, X, D, F} form a partition of the sample space to show that the average on-time arrival probability $P(T|U)$ for United flights is given by

$$P(T|U) = P(T|UC)P(C|U)+P(T|UL)P(L|U)+P(T|UX)P(X|U)+P(T|UD)\,P(D|U)+P(T|UF)P(F|U)$$

where $P(C|U)$ is the conditional probability that the flight is landing at Chicago given that it is a United flight etc. State a similar expression for $P(T|W)$. (cf. Ross pp. 98-99)

(c) 60% of United Airlines flights land at its hub (snowy Chicago), 15% at each of LA and San Francisco, and 5% at each of Phoenix and San Diego. 75% of America West flights land at **its** hub (sunny Phoenix), 10% at LA, and 5% at each of the other three airports. Use these numbers in the formula of part (b) and show that $P(T|U) < P(T|W)$, i.e., United has a worse average on-time performance even though it beats America West at all the five airports! Write a short explanation of the discrepancy between the per-airport on-time performance and the overall on-time performance.

3. [“…From the town of Bedrock, They’re a page right out of history…”] Fred suggests that he and Wilma play a game in which they will take turns tossing a fair coin; the first one to toss a

Head wins. Fred proposes that he will toss first. Wilma agrees to this, but, having taken ECE 313, she knows that she is at an disadvantage. So, she demands that in succeeding games, the *loser* of the previous game gets to toss first. For $n \ge 1$, let p_n and $q_n = 1 - p_n$ respectively denote the probabilities that Fred and Wilma win the n-th game. We saw in class that $p_1 = 2/3 > q_1 = 1/3$,

(a) Use the theorem of total probability to show that $p_2 = 4/9 < q_2 = 5/9$. More generally, show that for $n \ge 2$, $p_n = (2/3) - (1/3)p_{n-1}$ and $q_n = (2/3) - (1/3)q_{n-1}$ and use these difference equations to find p_n and q_n in terms of p_1 and q_1.
(If you never learned in Math 285/286 or Math 315 (or ECE 310) how to solve difference equations, assume that **for all values of n,** p_n can be expressed as $a + b\alpha^n$. Substitute into the above difference equation and solve for α and a; the value of b is obtained from the "initial condition" $p_1 = 2/3$. Repeat for q_n — it is the same difference equation but the "initial condition" $q_1 = 1/3$ is different.)

(b) What is the limit as $n \to \infty$ of p_n and q_n ? Is this game asymptotically fair?

(c) Fred now proposes that instead of the first one to toss a head winning the game, the first one who *matches* the previous toss (whether the previous toss is part of the current game or the last toss of the previous game) wins. Wilma accepts but generously insists that, as before, Fred still toss first (so that the poor schmuck has no previous toss to match on his first toss!). What are p_1 and q_1 now? p_2 and q_2? p_3 and q_3? Is this game asymptotically fair? Assume as before that the loser of each game tosses first in the next game.

4. ["It a'in't about bipartisan politics; it's about …"] The Senate of a certain country has 100 members consisting of 43 Conservative Republicans, 21 Conservative Democrats, 12 Liberal Republicans, and 24 Liberal Democrats. Before each vote, the groups caucus separately. Each group decides *independently* of the other groups whether to support or oppose the motion. *All* members of the group then vote in accordance with the caucus decision.
For those who think that this is the way politics works, I have this beautiful skyscraper on Wacker Drive in Chicago that I am willing to sell to you at a bargain price…

(a) Let A, B, C, and D respectively denote the events that the four groups vote for a spending plan that will lead to a 50% increase in the DoD budget over the next two years. Suppose that the probabilities of these independent events are P(A) = 0.9, P(B) = 0.6 P(C) = 0.5 and P(D) = 0.2. What is the probability that the bill passes?

(b) The President vetoes the bill as a budget-breaker. Let E, F, G, and H respectively denote the independent events that the four groups support the motion to override the veto. If these events have probabilities P(E) = 0.99, P(F) = 0.4, P(G) = 0.6, and P(H) = 0.1, what is the probability that the motion to override the veto passes ?
Political innocents are reminded that a simple majority (51 or more votes) is required to pass a bill, and a two-thirds majority (67 or more votes) to override a veto.

5. ["Tennis is a very simple game in theory: all you have to do is hit the ball over the net one more time than your opponent…"] Consider the following simplified model for a game of tennis. On each serve, let p denote the probability that player A wins the point, and $q = 1-p$ the probability that player B wins the point. Assume that the outcome of each serve is independent of all others. Player A wins the game if the score reaches 4–0, 4–1, or 4–2, while B wins the game if the score reaches 2–4, 1–4, or 0–4. Else, the score reaches 3–3 (called deuce) and from this point onwards, the game continues until one player is two points ahead of the other, and thereby wins the game.

(a) Find the probabilities that the score reaches 4–0, 4–1, or 4–2 and the probabilities that the score reaches 2–4, 1–4, or 0–4. I need 6 answers here!

(b) Find P(score reaches deuce). Show that the sum of the seven probabilities obtained in parts (a) and (b) is 1 regardless of the value of p.

(c) Given that the score is deuce, what is P(A wins the next two points)? (This means A wins the game). What is P(B wins the next two points)? (This means B wins the game). What is the probability that both players win one point each? In this case, the score is tied again, and is also called deuce.

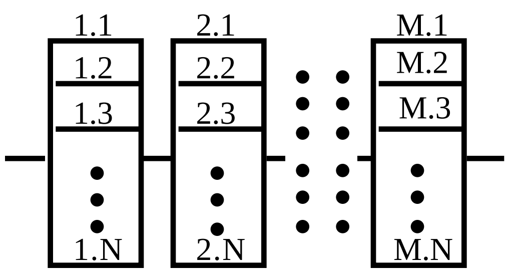
(d) Once the score reaches deuce, there *may* be further deuces until ultimately, either A or B wins both points and thereby wins the game. What is the probability that A ultimately wins the game given that the score is deuce? What is the probability that B ultimately wins the game given that the score is deuce? (Hint: these answers are different from those of part (c)) What is the probability that the game goes on forever with the score continuing to reach deuce after every two points?

(e) Use the results of parts (a)–(d) to express the probability that A wins the game as a function f(p) of p. A little thought shows that B wins with probability f(q) = f(1–p). Now, if p = 0, A wins no points which makes it difficult for him to win any games. Does your function f(p) satisfy f(0) = 0? If not, what does your f(p) give as the probability that A wins a game while losing every point? Similarly, if A wins every point, he is sure to win the game. Does your function f(p) satisfy f(1) = 1? If not, what does your f(p) give as the probability that A loses a game while winning every point?
Other reasonable properties of f(p) are f(0.5) = 0.5, f(p) + f(1–p) =1. Which of these is satisfied by your function f(p)?

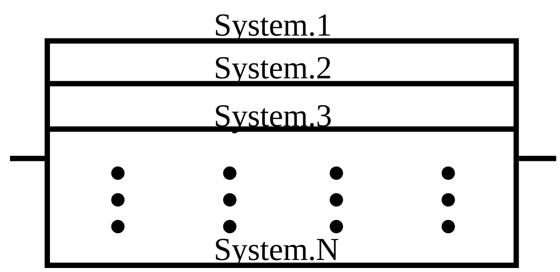
(f) Expand f(p) in a Taylor series in the neighborhood of p = 0.5 (only the first two terms are needed) What does this say about the probability of winning a game if $p = 0.5 + \varepsilon$ where ε is very small?

(g) Use your favorite graphing program to sketch f(p) as a function of p for $0 \leq p \leq 1$. Determine the minimum value of p for which $f(p) \geq 2/3$.

6. A system works if and only if all of its M components (numbered 1 through M) work. Each component fails (independently) with probability p. Consider two possible means of obtaining a more reliable system. We can replicate each component N times as shown in the graph model on the left. Or, we can replicate the entire system N times as shown in the graph model on the right. In either case, the result is called a replicated system. Note that both methods use the *same* number N of each component .



N-ary replication of each component



N-ary replication of entire system

As a more concrete example of the question, consider which of the following two methods provides more reliable transportation

- a single gigantic car with N engines, N transmissions, N brakes, … etc. that works (i.e. provides us with transportation) as long as at least one of its engines and at least one of its transmissions, and at least one of its brakes … works.
- N separate ordinary cars that fail as soon as any one of their components fail, but which together provide us with transportation as long as at least one car is in working condition.

(a) For each model, find the probability of replicated system failure in terms of p, N and M.

(b) Suppose that M = 5 and p = 0.2. If it is desired that the replicated system failure probability be less than 0.001, what should N be in each case?

(c) Repeat part (b) assuming that M = 1000.