Assigned: Wednesday, April 30
Due: Wednesday, May 7
Reading: Yates and Goodman: Chapters 5, 7 and 9.4–9.6
Problems:

1. Let $\Phi_{\mathbf{X}}(\omega)$ denote the characteristic function of a random variable $\mathbf{X}$.

(a) What is the value of $\Phi_{\mathbf{X}}(0)$? Does it matter whether $\mathbf{X}$ is discrete or continuous?

(b) Assume that $\Phi_{\mathbf{X}}(\omega)$ is differentiable at $\omega = 0$. What is the value of $\frac{d}{d\omega}\Phi_{\mathbf{X}}(\omega)$ at $\omega = 0$?

(c) What is the characteristic function of a Cauchy random variable? [Hint: the answer can be found in the inside front cover of the Kudeki-Munson Lecture Notes for ECE 210] Why can't the method of part (b) be used to compute the mean of a Cauchy random variable?

1.(a) $\Phi_{\mathbf{X}}(\omega) = E[\exp(j\omega\mathbf{X})]$ and hence $\Phi_{\mathbf{X}}(0) = E[\exp(j0\mathbf{X})] = E[1] = 1$ regardless of whether $\mathbf{X}$ is discrete or continuous.

(b) $\frac{d}{d\omega}\Phi_{\mathbf{X}}(\omega) = \frac{d}{d\omega}\int_{-\infty}^{\infty}\exp(j\omega u)\bullet f_{\mathbf{X}}(u)\,du = \int_{-\infty}^{\infty} ju\bullet\exp(j\omega u)\bullet f_{\mathbf{X}}(u)\,du$ and hence

$\left.\frac{d}{d\omega}\Phi_{\mathbf{X}}(\omega)\right|_{\omega=0} = \int_{-\infty}^{\infty} ju\bullet\exp(j0u)\bullet f_{\mathbf{X}}(u)\,du = j\int_{-\infty}^{\infty} u\bullet f_{\mathbf{X}}(u)\,du = jE[\mathbf{X}]$. A similar result holds for discrete random variables.

(c) The characteristic function of a Cauchy random variable is $\exp(-|\omega|)$, $-\infty < \omega < \infty$. Since it is not differentiable at 0, the method of part (b) does not apply.

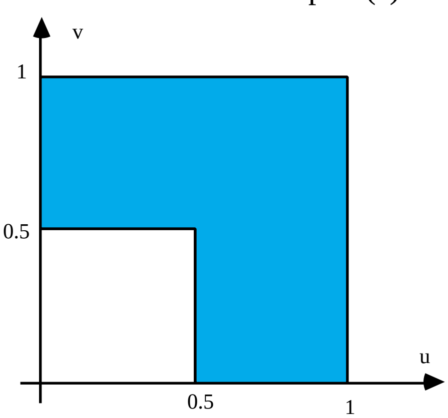
2. The random point $(\mathbf{X},\mathbf{Y})$ is uniformly distributed on the shaded region shown below.

(a) Find the marginal pdf $f_{\mathbf{X}}(u)$ of the random variable $\mathbf{X}$.

(b) *Write down* the marginal pdf $f_{\mathbf{Y}}(v)$ of the random variable $\mathbf{Y}$ from your answer to part (b).

(c) Find $P\{\mathbf{X} < \mathbf{Y} < 2\mathbf{X}\}$.

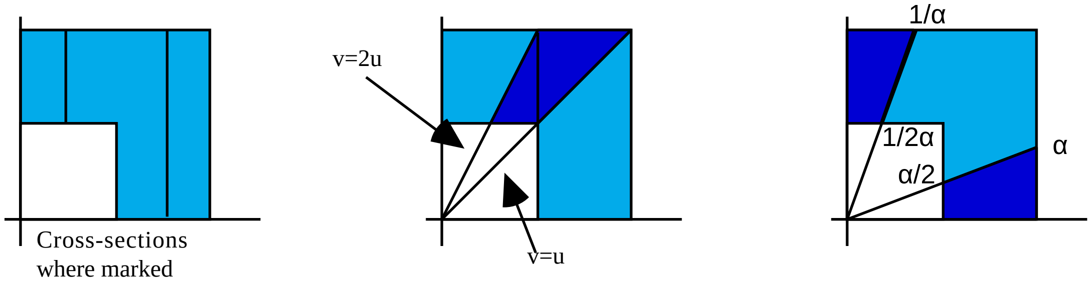
(d) What is $f_{\mathbf{X|Y}}(u|\alpha)$, the conditional pdf of $\mathbf{X}$ given that $\mathbf{Y} = \alpha$, if α satisfies $0 < \alpha < 1/2$? What is $f_{\mathbf{X|Y}}(u|\alpha)$, the conditional pdf of $\mathbf{X}$ given that $\mathbf{Y} = \alpha$, if α satisfies $1/2 < \alpha < 1$? Now, apply the theorem of total probability to compute the unconditional pdf of $\mathbf{X}$ from $f_{\mathbf{X|Y}}(u|\alpha)$. Do you get the same answer as in part (a)?



2.(a) The marginal pdf of $\mathbf{X}$ is the area of the cross-section of the joint pdf at the point u. There are two cases to be considered (0 < u < 1/2) and (1/2 < u < 1) as marked in the left-hand figure below. By inspection, we get that the area is $(4/3)\times(1/2) = 2/3$ for $0 < u < 1/2$, and $(4/3)\times 1 = 4/3$ for $1/2 < u < 1$, and 0 for all other u.

Thus, $f_{\mathbf{X}}(u) = \begin{cases} 2/3, & 0 < u < 1/2, \\ 4/3, & 1/2 < u < 1, \\ 0, & \text{elsewhere.} \end{cases}$ It is easily checked that the area under the pdf is 1.

(b) From the symmetry of the joint pdf, it is obvious that $f_{\mathbf{Y}}(v) = \begin{cases} 2/3, & 0 < v < 1/2, \\ 4/3, & 1/2 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$



(c) The probability desired is the volume in the darkly shaded region shown in the middle figure above. We can easily get the area of this region as the sum of two triangles $(1/2)\times(1/2)\times(1/4) + (1/2)\times(1/2)\times(1/2) = 3/16$ and the probability is $(4/3)\times(3/16) = 1/4$. Anti-segregationists (i.e. those who believe in integration) can get the same answer by writing $P\{\mathbf{X} < \mathbf{Y} < 2\mathbf{X}\} = \int_{v=1/2}^{1}\int_{u=v/2}^{v} 4/3\,du\,dv = \int_{v=1/2}^{1} 2v/3\,dv = v^2/3\Big|_{1/2}^{1} = \frac{1}{4}$.

(d) The conditional pdf of $\mathbf{X}$ given $\mathbf{Y} = \alpha$ is the cross-section $f_{\mathbf{X},\mathbf{Y}}(u,\alpha)$ of the joint pdf, normalized to have unit area. In this instance, it is obvious that the cross-section is a rectangle, and hence given $\mathbf{Y} = \alpha$, the conditional pdf of $\mathbf{X}$ is uniform on (1/2, 1) if $0 < \alpha < 1/2$, and uniform on (0, 1) if $1/2 < \alpha < 1$.

Thus, if $0 < \alpha < 1/2$, then
$$f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = \begin{cases} 2, & 1/2 < u < 1, \\ 0 & \text{otherwise,} \end{cases}$$
while if $1/2 < \alpha < 1$, then
$$f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = \begin{cases} 1, & 0 < u < 1, \\ 0 & \text{otherwise,} \end{cases}$$

The theorem of total probability gives $f_{\mathbf{X}}(u) = \int_0^1 f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) f_{\mathbf{Y}}(\alpha)\,d\alpha$

$= \int_0^{1/2} f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)(2/3)\,d\alpha + \int_{1/2}^{1} f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)(4/3)\,d\alpha$ on substituting the numerical value of $f_{\mathbf{Y}}(\alpha)\,d\alpha$ from part (b).

Now, for *fixed* u, $0 < u < 1/2$, $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = 0$ if $0 < \alpha < 1/2$, and $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = 1$ if $1/2 < \alpha < 1$. Hence, the two integrals are 0 and (1/2)•(4/3), giving that $f_{\mathbf{X}}(u) = 2/3$ for all numbers u, $0 < u < 1/2$. On the other hand, for *fixed* u, $1/2 < u < 1$, $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = 2$ if $0 < \alpha < 1/2$, and $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = 1$ if $1/2 < \alpha < 1$. Hence, the two integrals are (1/2)•2•(2/3) and (1/2)•(4/3), giving that $f_{\mathbf{X}}(u) = 4/3$ for all numbers u, $1/2 < u < 1$. This is the same answer as in part (b).

3. Let $E[\mathbf{X}] = 1$, $E[\mathbf{Y}] = 4$, $\text{var}(\mathbf{X}) = 4$, $\text{var}(\mathbf{Y}) = 9$, and $\rho_{\mathbf{X},\mathbf{Y}} = 0.1$

(a) If $\mathbf{Z} = 2(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})$, what is $E[\mathbf{Z}]$?

(b) If $\mathbf{T} = 2\mathbf{X}+\mathbf{Y}$ and $\mathbf{U} = 2\mathbf{X}-\mathbf{Y}$, what is $\text{cov}(\mathbf{T}, \mathbf{U})$?

(c) If $\mathbf{W} = 3\mathbf{X} + \mathbf{Y} + 2$, find $E[\mathbf{W}]$ and $\text{var}(\mathbf{W})$.

(d) If $\mathbf{X}$ and $\mathbf{Y}$ are jointly Gaussian random variables, and $\mathbf{W}$ is as defined in (c), what is $P\{\mathbf{W} > 0\}$?

3.(a) $E[\mathbf{Z}] = 2\{E[\mathbf{X}^2] - E[\mathbf{Y}^2]\} = 2\{\text{var}(\mathbf{X}) - \text{var}(\mathbf{Y}) + E^2[\mathbf{X}] - E^2[\mathbf{Y}]\} = -40$.

(b) $\text{cov}(\mathbf{T},\mathbf{U}) = \text{cov}(2\mathbf{X}+\mathbf{Y}, 2\mathbf{X}-\mathbf{Y}) = 4\cdot\text{var}(\mathbf{X}) - \text{var}(\mathbf{Y}) = 7$.

(c) $E[\mathbf{W}] = E[3\mathbf{X}+\mathbf{Y}+2] = 3E[\mathbf{X}] + E[\mathbf{Y}] + 2 = 9$.
$\text{var}(\mathbf{W}) = \text{var}(3\mathbf{X}+\mathbf{Y}+2) = \text{var}(3\mathbf{X}+\mathbf{Y})$ (why?) $= 9\cdot\text{var}(\mathbf{X})+\text{var}(\mathbf{Y})+6\cdot\text{cov}(\mathbf{X},\mathbf{Y}) = 9\cdot4+9+6\cdot(0.1\cdot2\cdot3) = 48.6$.

(d) $P\{\mathbf{W} > 0\} = 1 - \Phi((0-9/\sqrt{48.6}) = 1 - \Phi(-9/\sqrt{48.6}) = \Phi(9/\sqrt{48.6})$ (why?)

4. This problem has three independent parts. Do not apply the numbers from one part to the others.

(a) If $\text{var}(\mathbf{X} + \mathbf{Y}) = 36$ and $\text{var}(\mathbf{X} - \mathbf{Y}) = 64$, what is $\text{cov}(\mathbf{X}, \mathbf{Y})$? If you are also told that $\text{var}(\mathbf{X}) = 3\cdot\text{var}(\mathbf{Y})$, what is $\rho_{\mathbf{X},\mathbf{Y}}$?

(b) If $\text{var}(\mathbf{X} + \mathbf{Y}) = \text{var}(\mathbf{X} - \mathbf{Y})$, are $\mathbf{X}$ and $\mathbf{Y}$ uncorrelated ?

(c) If $\text{var}(\mathbf{X}) = \text{var}(\mathbf{Y})$, are $\mathbf{X}$ and $\mathbf{Y}$ uncorrelated ?

4.(a) $\text{var}(\mathbf{X}+\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) + 2\cdot\text{cov}(\mathbf{X},\mathbf{Y}) = 36$
$\text{var}(\mathbf{X}-\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) - 2\cdot\text{cov}(\mathbf{X},\mathbf{Y}) = 64$. Hence, $\text{cov}(\mathbf{X},\mathbf{Y}) = -7$.
If $\text{var}(\mathbf{X}) = 3\cdot\text{var}(\mathbf{Y})$, then $\text{var}(\mathbf{X}+\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) + 2\cdot\text{cov}(\mathbf{X},\mathbf{Y}) = 4\cdot\text{var}(\mathbf{Y}) - 14 = 36$, so that $\text{var}(\mathbf{Y}) = 12.5$, $\text{var}(\mathbf{X}) = 37.5$, and $\rho = \text{cov}(\mathbf{X},\mathbf{Y})/\sqrt{\text{var}(\mathbf{X})\text{var}(\mathbf{Y})} = -7/(12.5\sqrt{3})$.

(b) If var(**X**+**Y**) = var(**X**) + var(**Y**) + 2•cov(**X**,**Y**) = var(**X**–**Y**) = var(**X**) + var(**Y**) – 2•cov(**X**,**Y**), then it must be that cov(**X**,**Y**) = 0 and hence **X** and **Y** are uncorrelated.

(c) The values of var(**X**) and var(**Y**) do not affect the covariance computation at all. So, the random variables may or may not be uncorrelated.

5. Consider the random point (**X**,**Y**) of Problem 2 above.

(a) Compute E[**X**] and var(**X**).

(b) Explain why the random variable **Y** has the same mean and variance as **X**.

(c) Compute E[**XY**] and hence find cov(**X**,**Y**).
should hold. ***Is*** the above equation satisfied by the numerical values you obtained?

(d) The conditional pdf of **X** given **Y** = α was obtained in Problem 2 above, and it is easy to see that the conditional pdf of **Y** given **X** = α is similar. Now, the **best** (least mean-square error) estimate of **Y** given **X** = α is the mean of the conditional pdf of **Y** given **X** = α. Thus, if **X** has value $\alpha \le 0.5$, then $\hat{\mathbf{Y}}$, the best estimate of **Y**, is 0.75 while if **X** has value $\alpha > 0.5$, then $\hat{\mathbf{Y}} = 0.5$. Now, the **best linear** (least mean-square error) estimate of **Y** (given that **X** is known to have value α) is $\tilde{\mathbf{Y}} = a + b\alpha$ where a and b are given in Theorem 9.11 of Y&G. Compute a and b, and draw a graph showing the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ as functions of α. (Remember that $0 \le \alpha \le 1$). For what value(s) of α are the two estimates the same?

(e) Since the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ depend on the value of **X**, they really are *functions* of **X**, that is, they are *random variables* that can be expressed as $\hat{\mathbf{Y}} = \begin{cases} 0.75, & 0 \le \mathbf{X} \le 0.5, \\ 0.5, & 0.5 < \mathbf{X} \le 1 \end{cases}$ and $\tilde{\mathbf{Y}} = a + b\mathbf{X}$. What are the average and the mean-square errors of each estimate? That is, what are the values of $E[(\mathbf{Y} - \hat{\mathbf{Y}})]$, $E[(\mathbf{Y} - \tilde{\mathbf{Y}})]$, $E[(\mathbf{Y} - \hat{\mathbf{Y}})^2]$, and $E[(\mathbf{Y} - \tilde{\mathbf{Y}})^2]$?

5. It is easy to see that $f_{\mathbf{X}}(u)$ has constant value 2/3 for 0 < u < 1/2, and constant value 4/3 for 1/2 < v < 1.

(a) Hence, $E[\mathbf{X}] = \int_0^{1/2} u \cdot 2/3\, du + \int_{1/2}^{a} u \cdot 4/3\, du = \frac{(1/2)^2}{3} - \frac{0^2}{3} + \frac{2 \cdot 1^2}{3} - \frac{2 \cdot (1/2)^2}{3} = \frac{7}{12}$

and $E[\mathbf{X}^2] = \int_0^{1/2} u^2 \cdot 2/3\, du + \int_{1/2}^{a} u^2 \cdot 4/3\, du = \frac{2 \cdot (1/2)^3}{9} - \frac{2 \cdot 0^3}{9} + \frac{4 \cdot 1^3}{9} - \frac{4 \cdot (1/2)^3}{9} = \frac{5}{12}$.

Therefore, $\mathrm{var}(\mathbf{X}) = E[\mathbf{X}^2] - (E[\mathbf{X}])^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$.

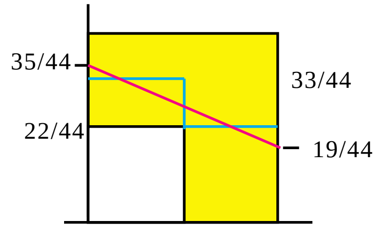
(b) Since the pdf of **Y** is the same as the pdf of **X**, it has the same mean and variance.

(c) $E[\mathbf{XY}] = \int_{u=0}^{1} \int_{v=0}^{1} uv \cdot 4/3\, dv\, du - \int_{u=0}^{12} \int_{v=0}^{1/2} uv \cdot 4/3\, dv\, du = \frac{4}{3}\left[\frac{1}{2} \times \frac{1}{2} - \frac{1}{8} \times \frac{1}{8}\right] = \frac{5}{16}$ (whoa! why?)

Hence, $\mathrm{cov}(\mathbf{X},\mathbf{Y}) = E[\mathbf{XY}] - E[\mathbf{X}]E[\mathbf{Y}] = \frac{5}{16} - \frac{49}{144} = \frac{-4}{144} = \frac{-1}{36}$ and thus $\rho = \frac{\mathrm{cov}(\mathbf{X},\mathbf{Y})}{\sqrt{\mathrm{var}(\mathbf{X})\mathrm{var}(\mathbf{Y})}} = \frac{-4}{11}$.

(d) $b = \rho \cdot \sqrt{\mathrm{var}(\mathbf{Y})/\mathrm{var}(\mathbf{X})} = \rho = \frac{-4}{11}$. $a = E[\mathbf{Y}] - bE[\mathbf{X}] = \frac{7}{12}\left[1 + \frac{4}{11}\right] = \frac{35}{44}$. Hence,

$\tilde{\mathbf{Y}} = \frac{35}{44} - \frac{4\mathbf{X}}{11} = \frac{35 - 16\mathbf{X}}{44}$ which is a straight line from (0,35/44) to (1,19/44). The graphs of the best estimate (a step function) and the best linear (straight line) estimate are as shown. Notice that the straight line estimate passes through the mean point (7/12,7/12) = (E[**X**],E[**Y**]) while the step function estimate does not. This is a characteristic of best linear estimators: if **X** equals its mean, the best linear estimate of **Y** is the mean of **Y**. Not always so for nonlinear optimum estimates. Notice also that the straight line estimate is a better approximation to the step function estimate in the interval (1/2, 1) because that is where most of the probability mass is!



The two estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ are the same at $\mathbf{X} = 1/8$ and $\mathbf{X} = 13/16$.

(e) Since $\hat{\mathbf{Y}}$ is a function of $\mathbf{X}$ (not $\mathbf{Y}$!) having value 3/4 if $\mathbf{X} < 1/2$ and value 1/2 if $\mathbf{X} > 1/2$, LOTUS allows us to write $E[(\mathbf{Y} - \hat{\mathbf{Y}})] = E[\mathbf{Y}] - E[\hat{\mathbf{Y}}] = \frac{7}{12} - \int_0^{1/2} (3/4)\,(2/3)\,du - \int_{1/2}^{1} (1/2)\,(4/3)\,du = \frac{7}{12} - \frac{1}{4} - \frac{1}{3} = 0$ so that the average error of the best mean-square estimate is 0. That $E[\mathbf{Y}] = E[\hat{\mathbf{Y}}]$ should be expected from Theorem 4.1 on p. 338 of the book. Similarly,

$$E[(\mathbf{Y} - \tilde{\mathbf{Y}})] = E[\mathbf{Y}] - E[\tilde{\mathbf{Y}}] = E[\mathbf{Y}] - E\left[\frac{35 - 16\mathbf{X}}{44}\right] = E[\mathbf{Y}] - \frac{35}{44} - \frac{16}{44} \times E[\mathbf{X}] = \frac{7}{12} - \frac{35}{44} - \frac{16}{44} \times \frac{7}{12} = 0.$$

On the other hand, $E[(\mathbf{Y} - \hat{\mathbf{Y}})^2] = \int_{v=1/2}^{1} \int_{u=0}^{1/2} (v-3/4)^2 \bullet 4/3\,du\,dv + \int_{v=0}^{1} \int_{u=1/2}^{1} (v-1/2)^2 \bullet 4/3\,du\,dv$

$= (2/3) \int_{v=1/2}^{1} (v-3/4)^2\,du\,dv + (2/3) \int_{v=0}^{1} (v-1/2)^2\,du\,dv = \frac{1}{16} = (0.25)^2$ while

$$E[(\mathbf{Y}-\tilde{\mathbf{Y}})^2] = E[(\mathbf{Y}-a-b\mathbf{X})^2] = \mathrm{var}(\mathbf{Y})\bullet[1-\rho^2] = \left(\frac{11}{144}\right)\left[1 - \left(\frac{-4}{11}\right)^2\right] = \frac{35}{528} \approx (0.257\ldots)^2 > \frac{1}{16} = (0.25)^2.$$

Ignoring $\mathbf{X}$ and estimating $\mathbf{Y}$ as $E[\mathbf{Y}] = \frac{7}{12}$ leads to mean-square error $\mathrm{var}(\mathbf{Y}) = \frac{11}{144} \approx (0.276\ldots)^2$