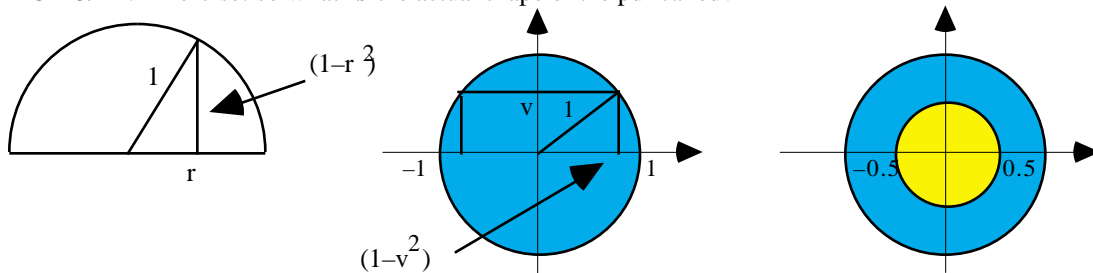


1.(a) $P\{R \leq r\}$ is the double integral of the joint pdf over the disc of radius r . Changing to polar coordinates, we get $P\{R \leq r\} = \int_0^r \int_0^{2\pi} g(r) r \, d\theta \, dr = 2 \int_0^r g(r) r \, dr$.

(b) Since only the upper limit depends on r and has derivative $= 1$, we get that $\frac{d}{dr}P\{R \leq r\}$ is the integrand evaluated at the upper limit, viz. $2 \cdot g(r)$ for $r \leq 1$.

(c) For the uniform distribution on the unit disc, $g(r) = (1/\pi)$ for $0 \leq r \leq 1$. Hence, the pdf of R is $2/\pi$ for $0 \leq r \leq 1$, and 0 otherwise. It is easily shown that this is a valid pdf.

(d) $\sqrt{1-u^2-v^2}$ has value $\sqrt{1-r^2}$ at all points (u,v) on a circle of radius $r < 1$. Thus, as can be deduced from the left-hand figure below, the surface $\sqrt{1-u^2-v^2}$ is a hemisphere of radius 1, and thus has volume $2\pi/3$. Consequently, the pdf surface encloses a volume $2\pi/3 = 1$ between itself and the $u-v$ plane, giving that $C = 3/2$. Exercise: so what is the actual shape of the pdf called?

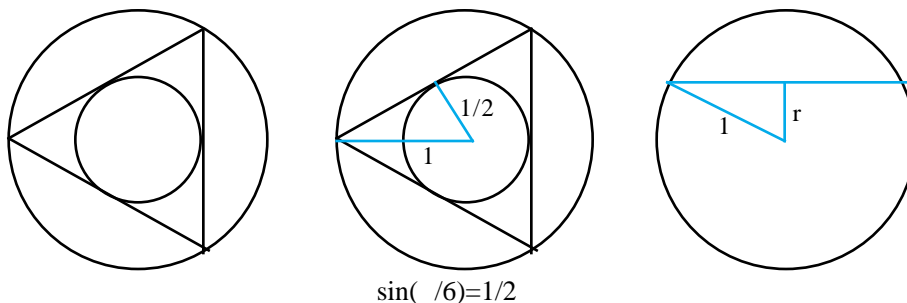


For those who cannot think geometrically but prefer to act integrally, a direct evaluation of the volume begins by setting up the integral via the middle figure above. For fixed v , $-1 < v < 1$, u varies between $-\sqrt{1-v^2}$ and $+\sqrt{1-v^2}$. The double integral is converted to polar coordinates and evaluated as shown below.

$$\int_{v=-1}^1 \int_{u=-\sqrt{1-v^2}}^{+\sqrt{1-v^2}} C \sqrt{1-u^2-v^2} \, du \, dv = \int_{r=0}^1 \int_{\theta=0}^{2\pi} C \sqrt{1-r^2} r \, d\theta \, dr = 2\pi C \int_0^1 \sqrt{1-r^2} r \, dr = -\frac{2\pi C}{3} (1-r^2)^{3/2} \Big|_0^1 = -\frac{2\pi C}{3} (1-1) + \frac{2\pi C}{3} (1-0) = \frac{2\pi C}{3}$$

giving $C = 3/2$ upon equating the volume to 1.

(e) Here, $g(r) = (3/2) \sqrt{1-r^2}$ and hence $P\{X^2 + Y^2 < 0.25\} = P\{R < 0.5\} = \int_0^{0.5} 2 \cdot r \cdot (3/2) \sqrt{1-r^2} \, dr$
 $= 3 \int_0^{0.5} r \sqrt{1-r^2} \, dr = -(1-r^2)^{3/2} \Big|_0^{0.5} = 1 - \frac{3\sqrt{3}}{8}$.



$$\sin(\pi/6) = 1/2$$

2.(a) As is obvious from the left-hand figure, the chord is longer than the side of the inscribed equilateral triangle if its midpoint lies inside the inscribed circle. It is easily shown that the radius of this inner circle is $1/2$. Hence, $P\{\text{random point lies inside inscribed circle}\} = \text{ratio of areas} = (1/2)^2 / (1)^2 = 1/4$.

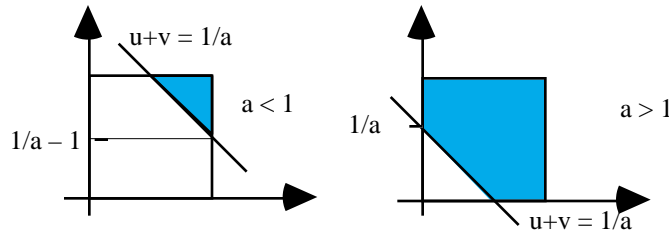
(b) If the midpoint is at distance r from the center of the circle, the chord has length $2\sqrt{1-r^2}$. Thus, we have that $L = 2\sqrt{1-R^2} = 2\sqrt{1-X^2-Y^2}$ where R is the distance of the random point (X,Y) from the origin. Now, note that $0 \leq L \leq 2$, and for any x , $0 \leq x \leq 2$, we have that $F_L(x) = P\{L \leq x\}$

$$= P\{2\sqrt{1-R^2} \leq x\} = P\{R \geq \sqrt{1-x^2/4}\} = 1 - P\{R \leq \sqrt{1-x^2/4}\} = 1 - \frac{(1-x^2/4)^{3/2}}{1^2} = x^2/4.$$

$$\text{Hence, } f_L(x) = \frac{d}{dx}F_L(x) = \begin{cases} x/2, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

(c) $E[L] = \int_0^2 x \cdot x/2 dx = 4/3.$

3. $I = 1/(R_1 + R_2)$ takes on values in the range $(1/2, \infty)$. Now, $F_I(a) = P\{I \leq a\} = P\{1/(R_1 + R_2) \leq a\} = P\{R_1 + R_2 \leq a^{-1}\}$. Now, $1 < a < \infty$ implies that $a^{-1} < 1$ while $1/2 < a < 1$ implies that $1 < a^{-1} < 2$. Thus, we have the two cases to consider as illustrated in the figure below.

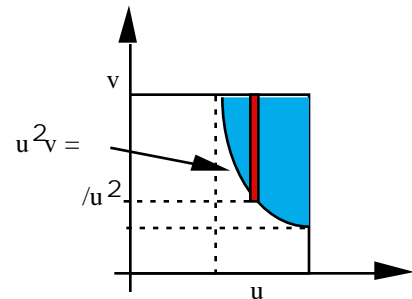


It follows readily from the diagrams that

$$F_I(a) = \begin{cases} 0, & a < 1/2, \\ (2 - a^{-1})^2/2, & 1/2 \leq a < 1, \\ 1 - a^{-2}/2, & 1 \leq a < \infty, \end{cases} \quad f_I(a) = \begin{cases} 2a^{-2} - a^{-3}, & 1/2 \leq a < 1, \\ a^{-3}, & 1 \leq a < \infty, \end{cases}$$

4. $Z = X^2 Y$. Then, $0 < Z < 1$, and for $0 < z < 1$,

$$\begin{aligned} P\{Z > z\} &= P\{X^2 Y > z\} \\ &= \int_{u=\sqrt{z}}^1 \int_{v=z/u^2}^1 2u \, dv \, du = \int_{u=\sqrt{z}}^1 2u(1 - z/u^2) \, du \\ &= u^2 - 2z \ln u \Big|_{\sqrt{z}}^1 = (1 - z) - (2z \ln 1 - 2z \ln \sqrt{z}). \end{aligned}$$



Hence, $1 - F_Z(z) = 1 - (1 - z) + 2z \ln \sqrt{z}$ for $0 < z < 1$, and the pdf $f_Z(z)$ is given by $-\ln z$, $0 < z < 1$, and 0 elsewhere. Quick check: Since $0 < z < 1$, $\ln z$ is negative, and hence $f_Z(z) > 0$ for $0 < z < 1$.

Furthermore, $F_Z(z) = -2z \ln \sqrt{z}$ approaches 1 as z approaches 1. In short, we have obtained a valid pdf.

5.(a) $f_{X^2}(v) = \frac{1}{2\sqrt{v}} \cdot [f_X(\sqrt{v}) + f_X(-\sqrt{v})] = \frac{1}{2\sqrt{v}} \cdot \frac{1}{\sqrt{2}} \cdot [\exp(-v/2) + \exp(-v/2)] = \frac{1}{\sqrt{2v}} \cdot \exp(-v/2)$

$= \frac{1}{\sqrt{2v}} \exp(-v/2) \cdot (v)^{1/2-1} / (1/2)$ where $\lambda = 1/2$ and $(1/2) = \sqrt{v}$ is a gamma pdf with parameters $(1/2, \lambda)$.

- (b) The sum of independent gamma random variables with parameters (t_i, λ) is a gamma random variable with parameter $(\sum t_i, \lambda)$. Since X^2 , Y^2 , and Z^2 are independent gamma random variables with parameter $(1/2, 1/2)$, $W = X^2 + Y^2 + Z^2$ is a gamma random variable with parameter $(3/2, 1/2)$. Its pdf is

$$f_W(w) = \begin{cases} \frac{(1/2)^{3/2} \exp(-w/2)}{\Gamma(3/2)}, & w > 0, \\ 0, & w < 0. \end{cases} = \frac{1}{3\sqrt{2}} \exp(-w/2), \quad w > 0, \quad 0, \quad w < 0.$$

- (c) $E[W] = E[X^2 + Y^2 + Z^2] = E[X^2] + E[Y^2] + E[Z^2] = 3$ because X , Y , and Z are $N(0, 1)$ random variables and $E[X^2] = \text{var}(X) + \mu^2 = \text{var}(X) = 1$ etc. We can check this by noting that a gamma random variable with parameter (t, λ) has mean t/λ . Shameless integrators can write

$$E[W] = \int_0^\infty w \cdot \frac{1}{3\sqrt{2}} \exp(-w/2) dw = \frac{4}{\sqrt{2}} \int_0^\infty x^{3/2} \exp(-x) dx = \frac{4}{\sqrt{2}} \left(\frac{5}{2}\right) = \frac{4}{\sqrt{2}} \left(\frac{3}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = 3$$

on substituting $2^{-2}x$ for w and remembering that $\Gamma(s) = (s-1)\Gamma(s-1)$ and that $(1/2) = \sqrt{2}$.

(d) $E[H] = E[(1/2)mW] = (1/2)mE[W] = (3/2)m^2 = (3/2)kT$, i.e. $m^2 = kT/m$. The pdf of W can thus be

$$\text{expressed as } f_W(w) = \begin{cases} \left(\frac{kT}{m}\right)^{-3/2} \sqrt{\frac{m}{2}} \exp(-m^2/2kT), & w \geq 0, \\ 0, & w < 0, \end{cases} \quad \text{while the pdf of } H = (m/2)W \text{ is}$$

$$f_H(h) = \frac{2}{m} f_W(h/m) = \begin{cases} 2 \sqrt{\frac{m}{2}} \left(\frac{kT}{m}\right)^{-3/2} \exp(-h^2/kT), & h \geq 0, \\ 0, & h < 0, \end{cases} \quad \text{which is the Maxwell-Boltzmann pdf.}$$

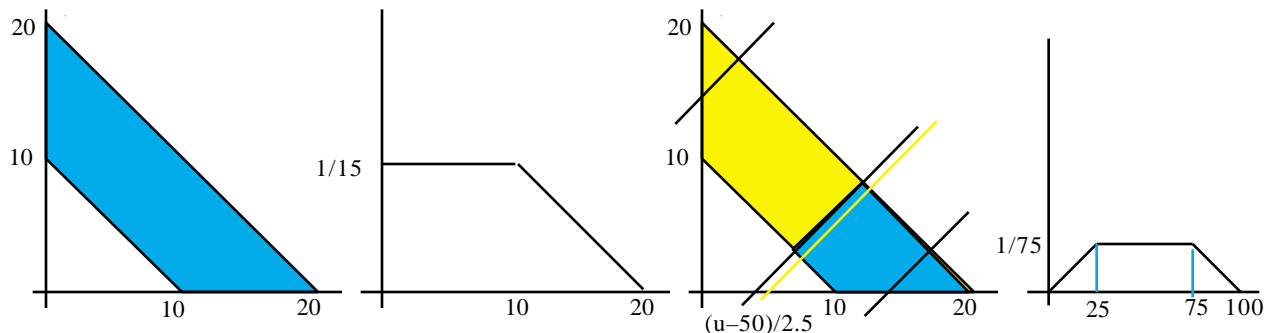
(e) $F_V(v) = P\{V \leq v\} = P\{\sqrt{W} \leq v\} = P\{W \leq v^2\} = F_W(v^2)$. Hence, $f_V(v) = \frac{d}{dv} F_V(v) = \frac{d}{dv} F_W(v^2) = 2v f_W(v^2)$ which gives the expression in the problem statement.

$$(f) \quad E[V] = \int_0^\infty v f_V(v) dv = \int_0^\infty v \cdot \frac{4}{\sqrt{2}} \left(\frac{m}{2kT}\right)^{3/2} v^2 \exp\left(-\frac{m}{2kT} v^2\right) dv = 2 \sqrt{\frac{2kT}{m}} \int_0^\infty x^3 \exp(-x^2) dx = 2 \sqrt{\frac{2kT}{m}} \quad (\text{Substitute } m^2/2kT = x)$$

$$\text{As a cross-check, } E[V] = E[\sqrt{W}] = \int_0^\infty \sqrt{w} \cdot \frac{1}{3} \sqrt{\frac{m}{2}} \exp(-m^2/2kT) dw = \frac{4}{\sqrt{2}} \int_0^\infty x^3 \exp(-x^2) dx = 2 \sqrt{\frac{2kT}{m}} \text{ on}$$

substituting $m^2/2kT = x$.

6.(a) The joint pdf has constant value K on the shaded region shown in the left-hand figure below. The area can be computed as the difference in area of two triangles as $(1/2) \cdot 20 \cdot 20 - (1/2) \cdot 10 \cdot 10 = 150$, giving $K = 1/150$.



(b) $f_R(x)$, the marginal pdf of R is the cross-sectional area of the joint pdf at x . Now, the cross-section is always a rectangle of height $1/150$, and its base is 10 if $0 < x < 10$, while the base is $20-x$ if $10 < x < 20$.

$$\text{Hence, we get that } f_R(x) = \begin{cases} 1/15, & 0 < x < 10, \\ (20-x)/150, & 10 < x < 20, \\ 0, & \text{elsewhere.} \end{cases}$$

Note that $P\{\text{student spends less than 10 hours studying for the ECE 313 Final}\} = 2/3$.

(c) T takes on values from 0 (when $S = 0$ and $R = 20$, that is, the student is sleepless and hopelessly confused) to 100 (when $S = 20$ and $R = 0$, that is, the student is well-rested and relies entirely on work done during the semester to carry him/her through the Final Exam). For any u , $0 < u < 100$,

$P\{T \leq u\} = P\{S + R \leq (u-50)/2.5\} = (1/150) \cdot \text{shaded trapezoidal area in third figure above}$, but note that there are special cases to be considered when $u < 25$ and $u > 75$. We can now proceed to find $P\{T \leq u\} = F_T(u)$ and differentiate to find the pdf, but an easier argument is as follows.

For $25 < u < 75$, and small Δu , $P\{u \leq T \leq u + \Delta u\} \approx f_T(u) \cdot \Delta u$

$= \text{volume between the two lines close together in the diagram} = (1/150) \cdot (10\sqrt{2}) \cdot (\Delta u \sqrt{2})$, since that thin sliver has height $(1/150)$, and the sides of the base are $10\sqrt{2}$ and $(u/2.5)\sqrt{2}$ respectively. We conclude that $f_T(u) = 1/75$ for $25 < u < 75$. A similar argument shows that the pdf of T increases linearly from 0 at $u = 0$ to $1/75$ at $u = 25$, and decreases from $1/75$ at $u = 75$ to 0 at $u = 100$. In short, it looks as shown above in the rightmost figure.