1.(a) A packet is received correctly if and only if all n bits are correct, which has probability $(1–p)^n = Q$. But, in case of errors, a re-transmisison can occur (upto a maximum of 5 tries). Thus, P{packet lost} = P{5 unsuccessful transmissions} = $(1–Q)^5$, and P{packet transmission is successful} = $1 – (1–Q)^5$.

(b) $\mathbf{X}_i$ takes on values 1, 2, 3, 4, 5 with probabilities Q, (1–Q)•Q, $(1–Q)^2$•Q, $(1–Q)^3$•Q, and $(1–Q)^4$ respectively. Note that although the first 4 probabilities are a geometric series, P{**X** = 5} is *not* the probability that a geometric random variable has value 5! If the first four transmissions are not received correctly, then **X** is *capped* at 5, and P{**X** = 5} = P{geometric random variable > 4} = $(1–Q)^4$.

(c)

$$
\begin{array}{llllll}
E[\mathbf{X}_i] & = 1\bullet Q & + 2\bullet(1–Q)\bullet Q & + 3\bullet(1–Q)^2\bullet Q & + 4\bullet(1–Q)^3\bullet Q & + 5\bullet(1–Q)^4 \\
 & = Q\bullet[1 & + (1–Q) & + (1–Q)^2 & + (1–Q)^3] & + (1–Q)^4 \\
 & + Q\bullet[& \;\;(1–Q) & + (1–Q)^2 & + (1–Q)^3] & + (1–Q)^4 \\
 & + Q\bullet[& & \;\;(1–Q)^2 & + (1–Q)^3] & + (1–Q)^4 \\
 & + Q\bullet[& & & \;\;(1–Q)^3] & + (1–Q)^4 \\
 & & & & & + (1–Q)^4
\end{array}
$$

$$= Q\bullet\frac{1 – (1–Q)^4}{1 – (1 – Q)} + (1–Q)^4 \quad + Q\bullet(1–Q)\bullet\frac{1 – (1–Q)^3}{1 – (1 – Q)} + (1–Q)^4$$

$$+ Q\bullet(1–Q)^2\bullet\frac{1 – (1–Q)^2}{1 – (1 – Q)} + (1–Q)^4 + \quad Q\bullet(1–Q)^3 \quad + (1–Q)^4$$

$$= 1 + (1–Q) + (1–Q)^2 + (1–Q)^3 + (1–Q)^4 = \frac{1 – (1–Q)^5}{1 – (1 – Q)} = \frac{1 – (1–Q)^5}{Q}$$

$$= \frac{1 – (1–5Q+10Q^2–10Q^3+5Q^4–Q^5)}{Q} = 5 – 10Q + 10Q^2 – 5Q^3 + Q^4$$

(d) P{all packets received successfully} = $[1 – (1–Q)^5]^L$. Note that the answer is *not* Q^L, which is the probability that all packets are received successfully on the first attempt.

2.(a) $P(A|B^c) = 1 – P(A^c|B^c) = 0.6$. $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.39$.
P(B|A) = P(AB)/P(A) = P(A|B)P(B)/P(A) = 0.21/0.39 = 7/13.

(b) P(F) = P(EF)/P(E|F) = P(F|E)P(E)/P(E|F) = (1/2) – (1/4)/(1/3) = 3/8.

(c) 1 ≥ P(G   H) = P(G) + P(H) – P(GH).
Hence, P(GH) ≥ 2/3 + 2/3 – 1 = 1/3, and P(G|H) = P(GH)/P(H) = P(GH)/(2/3) ≥ (1/3)/(2/3) = 1/2.

3. Given that **X** = n, we know that there was a success on the n-th trial, and r–1 successes in the previous n–1 trials so that $P\{\mathbf{X} = n\} = p\bullet\binom{n–1}{r–1}p^{r–1}(1–p)^{(n–1)–(r–1)} = \binom{n–1}{r–1}p^r(1–p)^{n–r}$. If there is a success on the i-th trial also, then there are r–2 successes on the *other* n–2 trials preceding the nth, i.e.
$P(\{\mathbf{X} = n\} \cap \{\text{success on i-th}\}) = p\bullet p\bullet\binom{n–2}{r–2}p^{r–2}(1–p)^{(n–2)–(r–2)} = \binom{n–2}{r–2}p^r(1–p)^{n–r}$. Hence, the conditional probability is the ratio of the binomial coefficients which works out to (r–1)/(n–1).
Alternatively, consider that given **X** = n, we know that there are r–1 successes in the previous n–1 trials. But, we saw in class that given that there are k successes on n trials, the probability of a success on the i-th trial is just k/n independent of p. So, here the answer must be (r–1)/(n–1) …

4. Let **Y** denote the number of passengers who show up for the flight, regardless of where they came from. When the connecting flight is on time, 15 passengers are guaranteed to show up. Of the other 90, the number who show up is a binomial random variable **Z** with parameters (90, 0.9). Thus, **Y** = 15 + **Z** when the connecting flight is on time. Otherwise, when the connecting flight is late, only **Z** passengers show up, and hence **Y** = **Z** in this case.

(a) Thus, P{**Y** ≤ 100 | on time} = P{**Z** ≤ 85} = 1 – P{**Z** > 85} = 0.95345... and P{**Y** ≤ 100 | late} = 1.

(b) Hence, P(**Y** ≤ 100) = (0.95345...)×(1/3) + 1×(2/3) = 0.98448.... Note that the only way for the airline to better the chances of everyone getting a seat is to increase the probability that the connecting flight is late! Finally, P{on time | **Y** ≤ 100} = P{**Y** ≤ 100 | on time}•P{on time}/P(**Y** ≤ 100) = 0.95345×(1/3)/0.98448 = 0.3226…

5.(a) $P(R_1) = 6/10$, $P(R_2|R_1) = 5/9$, $P(R_2|R_1^c) = 6/9$. Geez, that was an easy one!

(b) $P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|R_1^c)P(R_1^c) = (5/9)\bullet(6/10) + (6/9)\bullet(4/10) = 54/90 = 6/10$. Surprised?

(c) $P(R_1) = 6/10$ as before; $P(R_2|R_1) = 9/13$, $P(R_2|R_1^c) = 6/13$.

$P(R_2) = P(R_2|R_1)P(R_1)+P(R_2|R_1^c)P(R_1^c) = (9/13)\bullet(6/10) + (6/13)\bullet(4/10) = 78/130 = 6/10$. Surprised **now**?

A ball is being picked at random from an urn with 13 balls in it. It would seem that **all** probabilities would be of the form k/13, but instead we get an answer of 6/10. This is called Polya's urn scheme. Note that the answer is 6/10 if *any* number of additional balls of the same color are added; the result is not a magical property of 3. Furthermore, if the second ball is returned to the urn with 3 additional balls of the same color, the probability of the third ball being drawn being red is *still* 6/10. Weird, isn't it?

6. Let A and B denote respectively the events that your **first** choice and your **final** choice is the curtain concealing the prize. $P(A) = 1/3$, $P(A^c) = 2/3$.

(a) If you always switch, $P(B|A) = 0$, while $P(B|A^c) = 1$. Hence, $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 2/3$.

(b) If you never switch, then $P(B|A) = 1$, while $P(B|A^c) = 0$. Hence, $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 1/3$.

(c) If you decide at random, then $P(B|A) = P(B|A^c) = 1/2$ and $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 1/2$ also. Monty is correct in his assertion. (Would he lie to you? Besides, you saw it on **TV**, so it must be true!!!!)

7. This game is different from the one in Problem 6 in that *you have no idea what the rules of the game are.* If the man is intent on separating you from your money as quickly as possible, he will not offer the chance to switch unless you picked the shell hiding the pea in the first place! That is, if you picked the wrong shell, the man will reveal the pea and you will lose your bet. Of course, if you look like a person willing to play several rounds, the man may set you up by playing by Monty's rules (and allowing you to win with probability 2/3) for some time. Then you will place a large bet on the wrong shell, and all of a sudden, you will not be given the choice of changing your bet! Personally, I would stick with the shell originally chosen since it gives me at least a 1/3 probability of winning regardless of the man's strategy; your experience (and monetary losses) may vary

1 Since the pitcher can only throw fastballs, curveballs or sliders, $P(F) + P(C) + P(S) = 1$. Also, $P(C) = 2P(F)$ so we conclude that $P(S) = 1 – 3P(F)$. Now, $1/4 = P(H|C)P(C) + P(H|F)P(F) + P(H|S)P(S)$ $= P(H|C)2P(F)+ P(H|F)P(F) + P(H|S)(1 – 3P(F)) = P(F)[2/4 + 2/5 –3/6]$ which gives $P(F) = 5/24$, $P(C) = 10/24 = 5/12$, and $P(S) = 9/24 = 3/8$.

2.(a) $\mathbf{X}$ is a binomial random variable with parameters (10, 0.5) and mean $10 \bullet 0.5 = 5$.

(b) $P\{\mathbf{X} \geq 4\} = 1 – P\{\mathbf{X} \leq 3\} = 1 – 2^{-10}\left[1 + \binom{10}{1} + \binom{10}{2} + \binom{10}{3}\right] = 1 - \frac{176}{1024} = \frac{848}{1024} = \frac{53}{64}$.

$P\{\mathbf{X} \leq 5 \mid \mathbf{X} \geq 4\} = \frac{P\{4 \leq \mathbf{X} \leq 5\}}{P\{\mathbf{X} \geq 4\}} = 2^{-10}\left[\binom{10}{4} + \binom{10}{5}\right]/P\{\mathbf{X} \geq 4\} = \frac{462}{1024} \times \frac{1024}{848} = \frac{231}{424}$.

(c) $P\{\text{4th toss = Head} \mid \mathbf{X} = 4\}$

$$= \frac{P\{\text{4th toss = Head } \mathbf{and}\ \mathbf{X} = 4\}}{P\{\mathbf{X} = 4\}} = \frac{P\{\text{4th toss = Head } \mathbf{and} \text{ 3 Heads in other 9 tosses}\}}{P\{\mathbf{X} = 4\}}$$

$$= \frac{P\{\text{4th toss = Head}\}P\{\text{3 Heads in other 9 tosses}\}}{P\{\mathbf{X} = 4\}} \text{ (by independence of tosses) } = \frac{(1/2)\binom{9}{3}2^{-9}}{\binom{10}{4}2^{-10}} = \frac{4}{10}.$$

(d) For any arbitrary value of $P\{\text{Heads}\} = p > 0$, we have that $P\{\text{4th toss = Head} \mid \mathbf{X} = 4\}$

$$= \frac{P\{\text{4th toss = Head}\}P\{\text{3 Heads in other 9 tosses}\}}{P\{\mathbf{X} = 4\}} = \frac{p\binom{9}{3}p^3(1-p)^6}{\binom{10}{4}p^4(1-p)^6} = \frac{4}{10}.$$ Thus, not knowing p does

not disadvantage me; the probability is 4/10 regardless of the value of p. Now, a fair bet should be offering 3-to-2 odds, i.e. with a bet of \$1, you win \$1.50 roughly 40% of the time and lose \$1 roughly 60% of the time. A bookie who offered 2-to-1 odds would be losing \$0.20 per dollar bet and would soon be out of business, and perhaps wearing concrete overshoes as well! The fact that he knows the outcome of the 4th toss and yet is offering such great odds to induce you to bet that a Head occurred, leads to the suspicion that the only reason the bookie can afford to offer these odds is that the 4th toss resulted in a Tail, and thus is sure that he is not going to lose. I would **not** bet on a Head.

3.(a) 6 outcomes give a 3 on the first die and 6 more a 3 on the second die, but this double-counts (3,3). Thus, of the 11 outcomes in B_3, two (3,4) and (4,3) result in A, and hence $P(A|B_3) = 2/11$. Similarly for others.

(b) The theorem of total probability does not apply because the B_i are **not** mutually exclusive! In fact, $\sum P(B_i) = 11/6$, not 1, and the formula give an alleged value of $P(A) = 1/3$ which is twice as large. Note that $P(A|B_3)P(B_3) = P(AB_3)$ but the outcomes (3,4) and (4,3) in $P(AB_3)$ are also counted a second time as part of $P(AB_4)$ which explains why the computation gives a result twice as large.