1.(a) If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, then $f^{(n)}(x) = n!a_n$ = constant and all higher order derivatives are zero.

(b)-(d) True. $\frac{d}{dx}(1+x)^n = f^{(1)}(x) = n\bullet(1+x)^{n-1}$. $\frac{d^2}{dx^2}(1+x)^n = f^{(2)}(x) = n(n-1)\bullet(1+x)^{n-2}$, and more generally,

$f^{(k)}(x) = n(n-1)(n-2)\ldots(n-k+1)\bullet(1+x)^{n-k}$ for $1 \le k \le n$, while $f^{(k)}(x) = 0$ for $k > n$. Note that $f^{(n)}(x) = n!$

Hence, $f(x) = (1+x)^n = 1 + \frac{n}{1!}\bullet x + \frac{n(n-1)}{2!}\bullet x^2 + \ldots + \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}\bullet x^k + \ldots + \frac{n!}{n!}\bullet x^n$

$$= \binom{n}{0} + \binom{n}{1}\bullet x + \binom{n}{2}\bullet x^2 + \ldots + \binom{n}{k}\bullet x^k + \ldots + \binom{n}{n}\bullet x^n$$

(e) False. **(f)** $\frac{d}{dx}(1-x)^{-n} = f^{(1)}(x) = (-n)\bullet(1-x)^{-n-1}(-1) = n\bullet(1-x)^{-n-1}$.

$\frac{d^2}{dx^2}(1-x)^{-n} = f^{(2)}(x) = n(-n-1)\bullet(1-x)^{-n-2}(-1) = n(n+1)\bullet(1-x)^{-n-2}$, and more generally,

$f^{(k)}(x) = n(n+1)(n+2)\ldots(n+k-1)\bullet(1-x)^{-n-k}$ *for all $k > 0$*. Notice the difference between this and part (b).

Hence, the MacLaurin series for $(1-x)^{-n}$ is $1 + \frac{n}{1!}\bullet x + \frac{n(n+1)}{2!}\bullet x^2 + \ldots + \frac{n(n+1)(n+2)\ldots(n+k-1)}{k!}\bullet x^k + \ldots$

and *does not terminate* at degree n. The (n+1)-th term is $\frac{n(n+1)(n+2)\ldots(2n)}{(n+1)!}\bullet x^{n+1} = \frac{(n+2)\ldots(2n)}{(n-1)!}\bullet x^{n+1}$

(note the cancellation) while the (n+i)-th term is $\frac{n(n+1)\ldots(2n+i-1)}{(n+i)!}\bullet x^{n+i} = \frac{(n+i+1)\ldots(2n+i-1)}{(n-1)!}\bullet x^{n+i}$ where the numerator and denominator are each the product of n–1 consecutive integers.

(g) Setting n = 1 and 2 respectively in the above results, we get

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad \text{and} \qquad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

(h) When the MacLaurin series has infinitely many terms, we need to worry about convergence. For any given n, the absolute value of the ratio of the (n+i+1)-th term to the (n+i)-th termis $\frac{2n+i}{n+i}|x|$ which converges to $|x|$ as $i \to \infty$. The ratio test then says that the Taylor series for $(1-x)^{-n}$ converges if $|x| < 1$.

2.(a) $(1-x)^n = 1 - \frac{n}{1!}\bullet x + \frac{n(n-1)}{2!}\bullet x^2 + \ldots + (-1)^k\bullet\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}\bullet x^k + \ldots + (-1)^n\bullet\frac{n!}{n!}\bullet x^n$

$= \binom{n}{0} - \binom{n}{1}\bullet x + \binom{n}{2}\bullet x^2 + \ldots + (-1)^k\binom{n}{k}\bullet x^k + \ldots + (-1)^n\binom{n}{n}\bullet x^n$ while

$(1+x)^n = \binom{n}{0} + \binom{n}{1}\bullet x + \binom{n}{2}\bullet x^2 + \ldots + \binom{n}{k}\bullet x^k + \ldots + \binom{n}{n}\bullet x^n$

(b) The absolute values of the coefficients are the same in both expansions, but the even powers of x have positive signs in both expansions while the odd powers of x have opposite signs in the two expansions.

(c) The coefficient of x^{2k} in the series expansion of $(1+x)^n + (1-x)^n$ is $2\binom{n}{2k}$.

(d) $(1+1)^n + (1-1)^n = 2^n = 2\bullet\left[\binom{n}{0} + \binom{n}{2} + \ldots\right] = 2\bullet|S_{even}|$ giving that $|S_{even}| = 2^{n-1}$.

(e) Since there are a total of 2^n subsets and each belongs to either S_{even} or to S_{odd}, we see that $2^n = |S_{even}| + |S_{odd}|$, and since $|S_{even}| = 2^{n-1}$, it must be that $|S_{odd}| = 2^{n-1}$ also, no? Else, consider $h(x) = (1+x)^n - (1-x)^n$ and observe that $h(1) = 2^n = 2\bullet\left[\binom{n}{1} + \binom{n}{3} + \ldots\right] = 2\bullet|S_{odd}|$

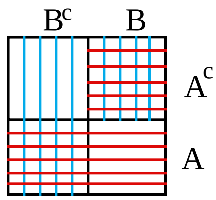
3.(a) Since each specialty flavor must contain at least one essence, the question is asking for the number of non-empty subsets of a set of 5 essences. The answer is $2^5 - 1 = 31$.

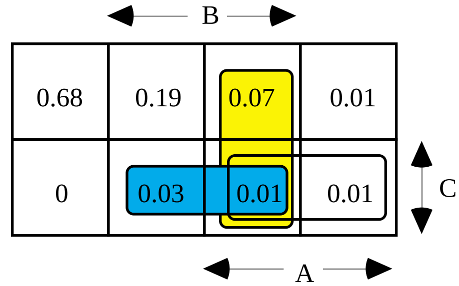
(b) In the diagram below, $A \cup B$, the event that at least one of A and B occurred, has horizontal stripes while $A^c \cup B^c$, the event that at least one of A and B did not occur, has vertical stripes. The union of these is obviously Ω. The forever plaid region is $A \triangle B = (A \cap B^c) \cup (A^c \cap B)$, the event that *exactly one* of A and B did not occur. Thus, we have that

$1 = P(\Omega) = P((A^c \cup B^c) \cup (A \cup B)) = P(A^c \cup B^c) + P(A \cup B) - P((A^c \cup B^c) \cap (A \cup B))$

$= P(A^c \cup B^c) + P(A \cup B) - P((A \cap B^c) \cup (A^c \cap B)) = P(A^c \cup B^c) + P(A \cup B) - P(A \triangle B)$.

Hence, $P(A \triangle B) = P(A^c \cup B^c) + P(A \cup B) - 1 = 0.4$.





4. This problem is best solved with a Venn diagram or more preferably a Karnaugh map as shown above . From the given values of P(ABC) = 0.01, P(AB) = 0.08, P(BC) = 0.04, and P(AC) = 0.02, it is easy to find $P(ABC^c)$ = 0.08 – 0.01 = 0.07, $P(A^cBC)$ = 0.04 – 0.01 = 0.03, and $P(AB^cC)$ = 0.02 – 0.01 = 0.01. Next, since P(A) = 0.1 = P(ABC) + $P(ABC^c)$ + $P(AB^cC)$ + $P(AB^cC^c)$, we get $P(AB^cC^c)$ = 0.01. Similarly, we can show that $P(A^cBC^c)$ = 0.19,and $P(A^cB^cC)$ = 0. Finally, we can get P(A B C) = P(ABC) + $P(ABC^c)$ + $P(A^cBC)$ + $P(AB^cC)$ + $P(AB^cC^c)$ + $P(A^cBC^c)$ + $P(A^cBC^c)$ = 0.32. Crosscheck: Proposition 4.4 gives P(A) + P(B) + P(C) – P(AB) – P(AC) – P(BC) + P(ABC) which is also 0.32. Finally, $P(A^cB^cC^c)$ = 1 – P(A B C) = 0.68. The probabilities asked for are

(a) P(only 1) = $P(AB^cC^c)$ + $P(A^cBC^c)$ + $P(A^cB^cC)$ = 0.01 + 0.19 + 0 = 0.2,

(b) P(at least 2) = P(AB BC AC) = P(ABC) + $P(ABC^c)$ + $P(A^cBC)$ + $P(AB^cC)$ = 0.01 + 0.07 + 0.03 + 0.01 = 0.12. Crosscheck: P(AB BC AC) = P(AB) + P(BC) + P(AC) – 2P(ABC)

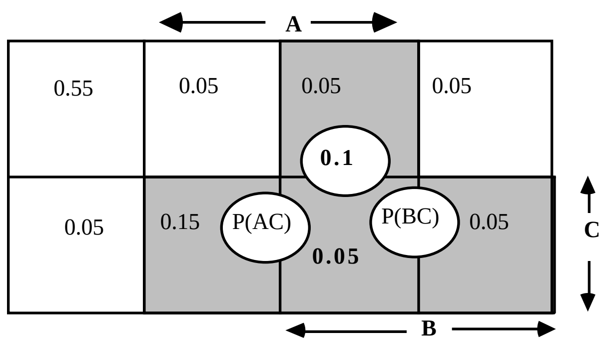
(c) P(B and at least one other) P(B ∩ (A C)) = P(AB) + P(BC) – P(ABC) = 0.08 + 0.04 – 0.01 = 0.11,

(d) P(no papers) $P(A^cB^cC^c)$ = 1 – P(A B C) = 0.68.

5.(a) A = (ddddddddd1: d = don't care if it is 0 or 1} giving |A| = 2^9 = 512; P(A) = $2^9/2^{10}$ = 1/2.

(b) The shift register contains 4 1's which can be chosen from the 10 positions in $\binom{10}{4} = \frac{10\times9\times8\times7}{1\times2\times3\times4} = 210$ ways (the remaining 6 positions have 0's in them.) Hence, B contains 210 outcomes and P(B) = $210/2^{10}$.

(c) AB is the set of shift register contents with 4 1's, one of which is in the least significant bit posiiton. The remaining 3 1's can be in any of $\binom{9}{3} = \frac{9\times8\times7}{1\times2\times3}$ = 84 positions. Hence, $P(A\cap B) = 84/2^{10}$;
P(A B) = P(A) + P(B) – $P(A\cap B)$ = $(512+210–84)/2^{10} = 638/2^{10}$;
P(A B) = P(A) + P(B) – 2$P(A\cap B)$ = $(512+210–168)/2^{10} = 554/2^{10}$.



P(AC) = 2P(BC)

Figure for Problem 6

6. We use the Venn diagram/Karnaugh map shown above.
P(A B) = P(A) + P(B) – P(AB) = 0.4, and hence $P(A^cB^c)$ = 1 – P(A B) = 0.6
P(AB AC BC) = 0.3 = P(AB)+P(AC)+P(BC)–2×P(ABC) = 0.1+3×P(BC)–2×0.05. Hence, P(BC) = 0.1,
P(AC) = 0.2, P(AC BC) = 0.2 + 0.1 – 0.05 = 0.25. Thus, $P(A^cB^cC)$ = P(snaps and crackles only)
= P(C) – P(AC BC) = 0.05, and $P(A^cB^cC^c)$ = P(Rice Krispies) = $P(A^cB^c)$ – $P(A^cB^cC)$ = 0.55.
$P(AB^cC^c)$ = P(A) – P(AB AC) = 0.2 – 0.1 – 0.1 + 0.05 = 0.05.
Similarly, $P(A^cBC^c)$ = P(B) – P(AB BC) = 0.3 – 0.1 – 0.2 + 0.05 = 0.05.