ECE 313 FINAL EXAMINATION

Friday May 9, 2003

Three hours

1. In a parallel universe far far away, the Parliament of a small island nation decided to change the form of its government from a constitutional hereditary monarchy to a constitutional appointed monarchy. The six applicants for the job of monarch: Andy, Beth, Chuck, Di, Eddie, and Fergie, are screened by a Search Committee that recommends a "short list" of three candidates to the full House. The House then selects one person from the short list to be the monarch. Assume that all possible short lists are equally likely to be recommended by the Search Committee.

(a) What is the probability that Chuck is on the short list?

(b) What is the probability that *at least one* woman is on the short list? Those wishing to pretend that they don't read *People* magazine are informed that Beth, Di, and Fergie are women; the rest are men.

The House selects the monarch from the short list using the following criteria:

(i) If Beth is on the short list, she is selected as the monarch.

(ii) If Beth is not on the short list, then the monarch is selected at random from the short list except that Di is never selected: if Di is on the short list, then one of the other two nonBethians on the short list is selected at random. [Hence the phrase "Never say Di!"]

(c) What is the probability that Chuck is selected as the monarch?

(d) Given that Chuck was selected as the monarch, what is the probability that Di was on the short list?

(e) Given that Chuck was selected as the monarch, what is the probability that the other two people on the short list were women?

2. Groucho, Chico, and Harpo decide who gets to woo Margaret Dumont in their next movie as follows. Each tosses a fair coin. If two of the coins show the same face and the third coin shows a different face, the tosser of the third gets to woo Margaret. Otherwise, it must be that all three coins are showing heads (or they all showing tails), in which case another round of coin tossing occurs.

(a) What is the probability that at least three rounds of tosses are required to make the decision?

(b) What is the probability that the decision is made on an even-numbered round?

(c) What is the probability that at least three rounds were required given that the decision was made on an even numbered round?

Chico is upset that Groucho has wooed Margaret in every movie thus far, and decides to improve his chances by secretly replacing his fair coin with a two-headed coin. The others continue to toss fair coins.

(d) What is the probability that Chico gets to woo Margaret? and has he improved his chances?

(e) Repeat part (d) under the further assumption that Harpo has spotted Chico's substitution and, being smarter than he looks, replaced *his* own coin with a two-tailed coin. Note only Groucho is tossing a fair coin now.

3. $\mathbf{X}$ denotes a continuous random variable whose probability density function (pdf) is of the form

$$f_{\mathbf{X}}(u) = \begin{cases} a, & 0 \le u < 1/2, \\ b, & 1/2 \le u \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$ where a and b are constants.

(a) What are the value(s) of a and b for which $\mathbf{X}$ has the smallest variance and what is this minimum variance?

(b) Express the value of $F_{\mathbf{X}}(3/4)$ in terms of a and b.

(c) If $E[\mathbf{X}] = 5/8$, what is the *numerical* value of $F_{\mathbf{X}}(3/4)$?

4. Let $\mathbf{X}$ denote the time of the first arrival after t = 0 in a Poisson process with arrival rate λ, and let $\mathbf{Y} = \exp(\mathbf{X})$. Find $f_{\mathbf{Y}}(v)$, the pdf of $\mathbf{Y}$.

5. The random point $(\mathbf{X},\mathbf{Y})$ is uniformly distributed on the region

$$\{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1,\ \min(u, v) \ge 1/2\}$$

(a) Find $P\{\mathbf{X}^2+\mathbf{Y}^2 < 1\}$. **(b)** What are the possible values of the random variable $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$?

(c) Find the pdf of $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$. **(d)** Show that your answer of part (c) is a valid pdf.

6. The random point $(\mathbf{X},\mathbf{Y})$ has joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} \frac{4}{\pi}, & u > 0,\ v > 0,\ u^2 + v^2 < 1, \\ 0, & \text{elsewhere} \end{cases}$

(a) Find $E[\mathbf{X}]$. **(b)** Find $E[\mathbf{X}^2]$ and hence determine var($\mathbf{X}$), the variance of $\mathbf{X}$.

Hint: It might be easier to first find $E[\mathbf{X}^2+\mathbf{Y}^2]$ and deduce the value of $E[\mathbf{X}^2]$ from this result.

(c) Find cov($\mathbf{X}$, $\mathbf{Y}$), the covariance of $\mathbf{X}$ and $\mathbf{Y}$.

7. The joint probability density function $f_{\mathbf{X},\mathbf{Y}}(u,v)$ for the continuous random variables $\mathbf{X}$ and $\mathbf{Y}$ has constant value on the region $\{(u, v) : 0 < u < 2,\ 0 < v < 2,\ 1 < u + v < 2$.

(a) Find $f_{\mathbf{X}}(u)$, the marginal probability density function for $\mathbf{X}$.

(b) What is $\hat{\mathbf{Y}}$, the minimum-mean-square-error (MMSE) estimator for $\mathbf{Y}$ given the value of $\mathbf{X}$? Your answer should be a function of $\mathbf{X}$. If you prefer, you can draw a neat graph of this function but be sure that the graph is labeled completely. Note that the problem is *not* asking for the *linear* MMSE estimator of $\mathbf{Y}$.