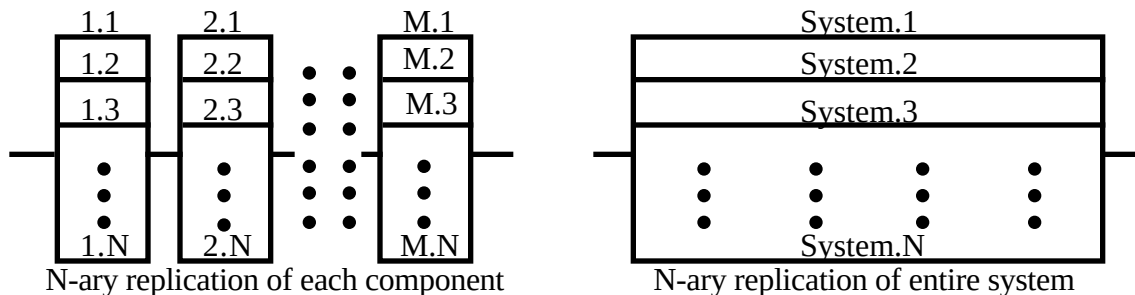


Assigned: Wednesday, March 6**Due:** Wednesday, March 13**Reading:** Ross, Chapter 4, Section 9; and Chapter 5**Noncredit Exercises:** Chapter 5: Problems 1-8; Theoretical Exercises: 1, 8**Note:** In honor of Engineering Open House, this Problem Set has only four problems on it...**Problems:**

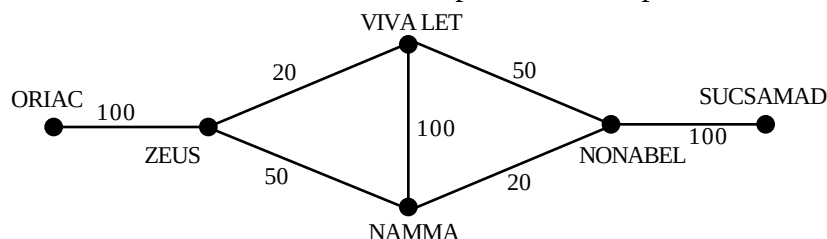
1. ["... It's not your father's Oldsmobile..."] A system works if and only if all of its M components (numbered 1 through M) work. Each component fails (independently) with probability p . Consider two possible means of obtaining a more reliable system. We can replicate each component N times as shown in the graph model on the left. Or, we can replicate the entire system N times as shown in the graph model on the right. In either case, the result is called a replicated system. Note that both methods use the *same* number N of each component.



A concrete example of the question we wish to consider is:

Which of the following two methods provides more reliable transportation?

- a single gigantic car with N engines, N transmissions, N brakes, ... etc. that works (i.e. provides us with transportation) as long as at least one of its engines and at least one of its transmissions, and at least one of its brakes ... works
 - N separate ordinary cars that fail as soon as any one of their components fail, but which together provide us with transportation as long as at least one car is in working condition.
- (a) For each model, find the probability of replicated system failure in terms of p , N and M .
- (b) Suppose that $M = 5$ and $p = 0.2$. If it is desired that the replicated system failure probability be less than 0.001, what should N be in each case?
- (c) Repeat part (b) assuming that $M = 1000$.
2. ["Reach out and touch someone"] MiddleEast Bell, a division of Psingular Corp., has built a telephone network as shown below. Terrorists attack each of the seven links. The attacks may be considered to be independent events, and the attack on a link succeeds in severing the link with probability p . If a link is severed, switches automatically re-route calls so as to avoid the failed link (if possible).
- (a) What is the probability of being able to call from ORIAC to SUCSAMAD?
- (b) Given that it is possible to call from ORIAC to SUCSAMAD, what is the conditional probability that the ZEUS to NAMMA link is in working condition?
- (c) The link capacities (i.e., the numbers of telephone calls that the links can carry (in either direction)) are as marked on the diagram. Let X denote the number of calls that can be made from ORIAC to SUCSAMAD. Find the pmf and the expected value of X .



3. The random variable X has the CDF shown in Figure 4.1 on p. 133 of Ross (5th edition) or Figure 4.7 on p. 168 of Ross (6th edition). Find $E[X]$.
4. The random variable X has probability density function

$$f_X(u) = \begin{cases} \alpha(1-u), & 0 < u < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Find $P\{6X^2 > 5X - 1\}$.
- (b) Find $F_X(u)$. Be sure to specify the value of $F_X(u)$ for all u .

Noncredit optional exercise:

Some of you were concerned about the use of the Chebyshev inequality in Problem 2 of Problem Set #5 to get that at least 4123 repetitions were needed to achieve the desired error probability. Here is a tighter result. With X denoting an (n, p) binomial random variable,

- (a) Show that if $\lambda > 0$, then $\exp\left(\lambda u - \frac{n\lambda^2}{2}\right) > 1$ for all $u > \frac{n}{2}$ and that therefore

$$P\{X > \frac{n}{2}\} = \sum_{u > n/2} p_X(u) < \sum_{u > n/2} \exp\left(\lambda u - \frac{n\lambda^2}{2}\right) p_X(u) < \sum_{u=0}^n \exp\left(\lambda u - \frac{n\lambda^2}{2}\right) p_X(u)$$

$$\text{i.e., } P\{X > \frac{n}{2}\} < E\left[\exp\left(\lambda X - \frac{n\lambda^2}{2}\right)\right]$$

This result is called a Chernoff bound (see also Chapter 8 of Ross).

- (b) Use LOTUS to prove directly from the known pmf of X that

$$E[\exp(\lambda X)] = (1 - p + p \cdot \exp(\lambda))^n$$

and hence that $P\{X > n/2\} < \exp(-n\lambda/2) \cdot (1 - p + p \cdot \exp(\lambda))^n$ for all values of $\lambda > 0$.

- (c) Evaluate the above bound on $P\{X > n/2\}$ at $\lambda = \ln 2$ and at $\lambda = -\ln p$.
- (d) Prove that as a function of λ , $\exp(-n\lambda/2) \cdot (1 - p + p \cdot \exp(\lambda))^n$ has minimum value $(2\sqrt{p(1-p)})^n$ at $\lambda = \ln((1-p)/p)$. How different is $\ln((1-p)/p)$ from $-\ln p$?
- (e) Repeat part (c) of Problem #2 of Problem Set #5 to choose the value of n using the Chernoff bound rather than the Chebyshev inequality.

Who gets the raise for designing an efficient communication system, you who slogged through to here or your pal who stopped at part (c) of Problem #2 of Problem Set #5?