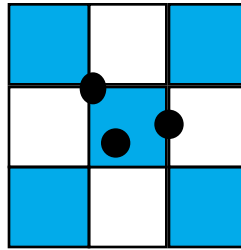


Assigned: Wednesday, April 24, 2002**Due:** Wednesday, May 1, 2002**Reading:** Ross, Chapter 7, Sections 1-5; Chapter 8 Sections 1-4**Noncredit Exercises:** Chapter 7: Problems 1, 16, 26, 29, 34, 36;

Theoretical Exercises: 1, 2, 17, 22, 23, 40; Chapter 8: Problems 1-9, 15

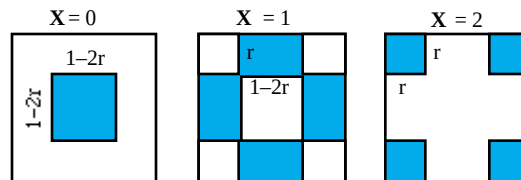
Problems:

1. In a carnival game, you place a bet and toss a circular coin of radius $r < 1$ onto a table whose top has a 3×3 "checkerboard" of unit squares. The object is to get the coin to land in the central square. Assume that the position of the center of the coin is uniformly distributed on the *central* square of the checkerboard, that is, you always manage to toss the coin onto the central square, even if it lies only partially inside the square.



If the coin intersects two lines or lies outside the central square, you lose the money you bet. If the coin intersects only one line, you get back the money you bet plus winnings equal to the amount that you bet. If the coin lies entirely inside the central square, you get back the money you bet plus winnings equal to three times the amount that you bet.

- (a) Let X denote the number of lines that your coin crosses. What is the pmf of X ?
- (b) If the game is fair, that is, the expected amount that you win is 0, what is the value of r ?
- 1.(a) If $r \geq 1/2$, the coin always intersects two lines and thus $P\{X = 2\} = 1$ if $r \geq 1/2$. Thus, if $r < 1/2$, we assume that the center of the coin is a bivariate random variable uniformly distributed on the unit square. Then, the cases $\{X = 0\}$, $\{X = 1\}$ and $\{X = 2\}$ are as illustrated below, where the event in question occurs if the coin's center lies in the shaded regions shown.



We easily see that $P\{X = 0\} = (1-2r)^2$, $P\{X = 1\} = 4 \cdot r \cdot (1-2r)$, $P\{X = 2\} = 4 \cdot r^2$, $0 \leq r \leq 1/2$.

(Sanity check: $((1-2r)^2 + 4 \cdot r \cdot (1-2r) + 4 \cdot r^2) = (1 - 4r + 4r^2) + (4r - 8r^2) + 4r^2 = 1$ for all choices of r .)

- (b) The *winnings* are 3, 1, and -1 corresponding to values 0, 1, 2 of X . Hence, the expected amount of money won is $3 \cdot (1-2r)^2 + 1 \cdot (4 \cdot r \cdot (1-2r)) + (-1) \cdot (4 \cdot r^2) = 3 - 8r$, and this is zero if $r = 3/8$.

2. An VLSI chip has constant hazard rate $\lambda = -\ln 0.999/\text{week}$.

(a) What is the average lifetime (in weeks)? What is the median lifetime?

(b) What is the probability that the module lasts for at least one week?

Now suppose that three identical chips are organized into a triple-modular-redundancy (TMR) system in which we assume that the majority-logic gate cannot fail. Furthermore, we assume that the three modules fail independently of one another, that is, their lifetimes X_1 , X_2 , and X_3 are independent random variables. Thus, the three events $\{X_1 > t_1\}$, $\{X_2 > t_2\}$, and $\{X_3 > t_3\}$ are independent for all possible values of t_1, t_2, t_3 .

Let Y denote the length of time for which the TMR system functions correctly.

- (c) If you are told that the event $\{Y > t\}$ occurred, what can you say about the occurrence (or nonoccurrence) of the events $\{X_1 > t\}$, $\{X_2 > t\}$, and $\{X_3 > t\}$?
- (d) Show that $P\{Y > t\} = 3\exp(-2\lambda t) - 2\exp(-3\lambda t)$ and use this result to find the average lifetime and the median lifetime of the TMR system. Compare your answers to those in part (a). Do the results surprise you? Is the TMR system improving performance the way it is alleged to?

- (e) What is the probability that the TMR system functions correctly for at least one week? Compare this answer to that of part (b). Do you think that the TMR system is more reliable or less reliable?
- (f) Find t such that $P\{Y > t\} = 0.999$ and compare the answer to that of part (b). Has the TMR system improved performance?
- 2.(a) The lifetime is an exponential random variable with parameter λ . Hence, the *average* lifetime is $\lambda^{-1} = (-\ln 0.999)^{-1} \approx 999.5$ weeks which is more than 19 years! The *median* lifetime is T where $P\{X > T\} = \exp(-\lambda T) = 1/2$, i.e. $T = \lambda^{-1} \ln 2 \approx 692.8$ weeks which is only 13 years and a few months.
- (b) $P\{X > 1\} = \exp(-\lambda) = 0.999$ so even with this long-lived module, we can only be 99.9% sure that the thing is going to last for a week!
- (c) $\{Y > t\}$ is the (disjoint) union of the four events $\{X_1 > t\} \cap \{X_2 > t\} \cap \{X_3 > t\}$, $\{X_1 > t\} \cap \{X_2 > t\} \cap \{X_3 \leq t\}$, $\{X_1 > t\} \cap \{X_2 \leq t\} \cap \{X_3 > t\}$, and $\{X_1 \leq t\} \cap \{X_2 > t\} \cap \{X_3 > t\}$.
- (d) Since $P\{X_i > t\} = \exp(-\lambda t)$, we readily find that the probability of the first event in (c) is $\exp(-3\lambda t)$, while the other three have probability $\exp(-2\lambda t) [1 - \exp(-\lambda t)]$. Hence, $P\{Y > t\} = 3 \exp(-2\lambda t) - 2 \exp(-3\lambda t)$.
- Furthermore, $E[Y] = \int_0^\infty P\{Y > t\} dt = \int_0^\infty 3 \exp(-2\lambda t) - 2 \exp(-3\lambda t) dt = (3/2\lambda) - (2/3\lambda) = (5/6\lambda) < 1/\lambda$
- so that the average lifetime of the TMR system is actually smaller than that of the individual modules! To find the median lifetime, we set $P\{Y > T\} = 3 \exp(-2\lambda T) - 2 \exp(-3\lambda T) = 3p^2 - 2p^3 = 1/2$. It is easy to show that $p = 1/2$ is a solution while the other two solutions correspond to $p < 0$ and $p > 1$. Hence, the median lifetime of the TMR system is also $\lambda^{-1} \ln 2 \approx 692.8$ weeks. Thus, in terms of these averages, the TMR system is just a waste of money; you have to pay more than three times as much (the majority-logic gate, not being one of the best things in life, is not free!) and the performance seems to be worse than that of a single module.
- (e) $P\{Y > 1\} = 3 \exp(-2\lambda) - 2 \exp(-3\lambda) = 3(0.999)^2 - 2(0.999)^3 = 0.999997002$ which is *much better* than the comparable figure for the single module.
- (f) $P\{Y > t\} = 0.999$ for $t = -\lambda^{-1} \ln 0.98163 \approx 18.53$ weeks which is more than 4 months! Thus, we can be 99.9% sure that the TMR system will work for at least 18.53 weeks. This is **much** longer than the week that a single module might be expected to last with 99.9% probability. It is in **these** matters that the TMR system makes a difference, and why it is one of the choices available to the system designer who wants to create reliable systems. As noted above, the average lifetime of the TMR system is worse than that of the single module. This is an illustration of the truism that complex systems have worse reliability than simple systems. What the TMR system is giving up is average lifetime in return for **much** higher reliability in early life.

3. Let $E[X] = 1$, $E[Y] = 4$, $\text{var}(X) = 4$, $\text{var}(Y) = 9$, and $\rho_{X,Y} = 0.1$

- (a) If $Z = 2(X+Y)(X-Y)$, what is $E[Z]$?
- (b) If $T = 2X+Y$ and $U = 2X-Y$, what is $\text{cov}(T, U)$?
- (c) If $W = 3X + Y + 2$, find $E[W]$ and $\text{var}(W)$.
- (d) If X and Y are jointly Gaussian random variables, and W is as defined in (c), what is $P\{W > 0\}$?
- 3.(a) $E[Z] = 2\{E[X^2] - E[Y^2]\} = 2\{\text{var}(X) - \text{var}(Y) + E^2[X] - E^2[Y]\} = -40$.
- (b) $\text{cov}(T, U) = \text{cov}(2X+Y, 2X-Y) = 4\text{var}(X) - \text{var}(Y) = 7$.
- (c) $E[W] = E[3X+Y+2] = 3E[X] + E[Y] + 2 = 9$.
 $\text{var}(W) = \text{var}(3X+Y+2) = \text{var}(3X+Y)$ (why?) $= 9\text{var}(X) + \text{var}(Y) + 6\text{cov}(X, Y) = 9 \cdot 4 + 9 + 6 \cdot (0.1 \cdot 2 \cdot 3) = 48.6$.
- (d) $P\{W > 0\} = 1 - \Phi((0-9)/\sqrt{48.6}) = 1 - \Phi(-9/\sqrt{48.6}) = \Phi(9/\sqrt{48.6})$ (why?)

4. This problem has three independent parts. Do not apply the numbers from one part to the others.

- (a) If $\text{var}(X + Y) = 36$ and $\text{var}(X - Y) = 64$, what is $\text{cov}(X, Y)$? If you are also told that $\text{var}(X) = 3\text{var}(Y)$, what is $\rho_{X,Y}$?
- (b) If $\text{var}(X + Y) = \text{var}(X - Y)$, are X and Y uncorrelated?
- (c) If $\text{var}(X) = \text{var}(Y)$, are X and Y uncorrelated?
- 4.(a) $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y) = 36$
 $\text{var}(X-Y) = \text{var}(X) + \text{var}(Y) - 2\text{cov}(X, Y) = 64$. Hence, $\text{cov}(X, Y) = -7$.
 If $\text{var}(X) = 3\text{var}(Y)$, then $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y) = 4\text{var}(Y) - 14 = 36$, so that $\text{var}(Y) = 12.5$, $\text{var}(X) = 37.5$, and $\rho = \text{cov}(X, Y) / \sqrt{\text{var}(X)\text{var}(Y)} = -7 / (12.5\sqrt{3})$.

- (b) If $\text{var}(\mathbf{X} + \mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) + 2 \cdot \text{cov}(\mathbf{X}, \mathbf{Y}) = \text{var}(\mathbf{X} - \mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) - 2 \cdot \text{cov}(\mathbf{X}, \mathbf{Y})$, then it must be that $\text{cov}(\mathbf{X}, \mathbf{Y}) = 0$ and hence $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated.
- (c) The values of $\text{var}(\mathbf{X})$ and $\text{var}(\mathbf{Y})$ do not affect the covariance computation at all. So, the random variables may or may not be uncorrelated.

5. Consider the random point $(\mathbf{X}, \mathbf{Y})$ of Problem 2 of Problem Set #13.

(a) Compute $E[\mathbf{X}]$ and $\text{var}(\mathbf{X})$.

(b) Explain why the random variable $\mathbf{Y}$ has the same mean and variance as $\mathbf{X}$.

(c) Compute $E[\mathbf{X}\mathbf{Y}]$ and hence find $\text{cov}(\mathbf{X}, \mathbf{Y})$.

(d) GLOTUS tells us that $E[g(\mathbf{X}, \mathbf{Y})] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u, v) \cdot f_{\mathbf{X}, \mathbf{Y}}(u, v) du dv$. Now, consider the

function $g(\mathbf{X}, \mathbf{Y}) = \min\{\mathbf{X}, \mathbf{Y}\}$. Use GLOTUS to find $E[g(\mathbf{X}, \mathbf{Y})] = E[\min\{\mathbf{X}, \mathbf{Y}\}]$ by showing that the global integrand $g(u, v) \cdot f_{\mathbf{X}, \mathbf{Y}}(u, v)$ can be expressed as

$u \cdot f_{\mathbf{X}, \mathbf{Y}}(u, v)$ for all points (u, v) in the plane for which $u < v$

and as $v \cdot f_{\mathbf{X}, \mathbf{Y}}(u, v)$ for all points (u, v) in the plane for which $v \leq u$.

Thus, the global integral can be expressed as the sum of integrals over two disjoint regions (these are divided by the line of slope 1 through the origin) of the plane.

Reminder: In case you got lost in the above verbiage, you are to find $E[\min\{\mathbf{X}, \mathbf{Y}\}]$.

(e) Repeat part (d) to find $E[h(\mathbf{X}, \mathbf{Y})] = E[\max\{\mathbf{X}, \mathbf{Y}\}]$.

(f) Compare the numerical values that you obtained in parts (d) and (e), and state whether or not $E[\max\{\mathbf{X}, \mathbf{Y}\}]$ is larger than $E[\min\{\mathbf{X}, \mathbf{Y}\}]$.

Should $E[\max\{\mathbf{X}, \mathbf{Y}\}]$ exceed $E[\min\{\mathbf{X}, \mathbf{Y}\}]$ (even if, according to your computed values, it does not in this instance)?

(g) Since $\min\{\mathbf{X}, \mathbf{Y}\} + \max\{\mathbf{X}, \mathbf{Y}\} = \mathbf{X} + \mathbf{Y}$, the following equation

$$E[\min\{\mathbf{X}, \mathbf{Y}\}] + E[\max\{\mathbf{X}, \mathbf{Y}\}] = E[\mathbf{X}] + E[\mathbf{Y}]$$

should hold. **Is** the above equation satisfied by the numerical values you obtained?

(h) The conditional pdf of $\mathbf{X}$ given $\mathbf{Y} = \alpha$ was obtained in Problem 2 of Problem Set #13, and it is easy to see that the conditional pdf of $\mathbf{Y}$ given $\mathbf{X} = \alpha$ is similar. Now, the **best** (least mean-square error) estimate of $\mathbf{Y}$ given $\mathbf{X} = \alpha$ is the mean of the conditional pdf of $\mathbf{Y}$ given $\mathbf{X} = \alpha$. Thus, if $\mathbf{X}$ has value $\alpha \leq 0.5$, then $\hat{\mathbf{Y}}$, the best estimate of $\mathbf{Y}$, is 0.75 while if $\mathbf{X}$ has value $\alpha > 0.5$, then $\hat{\mathbf{Y}} = 0.5$. Now, the **best linear** (least mean-square error) estimate of $\mathbf{Y}$ (given that $\mathbf{X}$ is known to have value α) is $\tilde{\mathbf{Y}} = a + b\alpha$ where a and b are given in Eq. (5.4) of Chapter 7 of Ross. Compute a and b , and draw a graph showing the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ as functions of α . (Remember that $0 \leq \alpha \leq 1$). For what value(s) of α are the two estimates the same?

(i) Since the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ depend on the value of $\mathbf{X}$, they really are *functions* of $\mathbf{X}$, that is, they are *random variables* that can be expressed as $\hat{\mathbf{Y}} = \begin{cases} 0.75, & 0 \leq \mathbf{X} \leq 0.5, \\ 0.5, & 0.5 < \mathbf{X} \leq 1 \end{cases}$ and

$\tilde{\mathbf{Y}} = a + b\mathbf{X}$. What are the average and the mean-square errors of each estimate? That is, what are the values of $E[(\mathbf{Y} - \hat{\mathbf{Y}})]$, $E[(\mathbf{Y} - \tilde{\mathbf{Y}})]$, $E[(\mathbf{Y} - \hat{\mathbf{Y}})^2]$, and $E[(\mathbf{Y} - \tilde{\mathbf{Y}})^2]$?

5. It is easy to see that $f_{\mathbf{X}}(u)$ has constant value $2/3$ for $0 < u < 1/2$, and constant value $4/3$ for $1/2 < u < 1$.

$$(a) \text{ Hence, } E[\mathbf{X}] = \int_{1/2}^1 u \cdot 2/3 du + \int_0^{1/2} u \cdot 4/3 du = \frac{(1/2)^2}{3} - \frac{0^2}{3} + \frac{2 \cdot 1^2}{3} - \frac{2 \cdot (1/2)^2}{3} = \frac{7}{12}$$

$$\text{and } E[\mathbf{X}^2] = \int_0^{1/2} u^2 \cdot 2/3 du + \int_{1/2}^1 u^2 \cdot 4/3 du = \frac{2 \cdot (1/2)^3}{9} - \frac{2 \cdot 0^3}{9} + \frac{4 \cdot 1^3}{9} - \frac{4 \cdot (1/2)^3}{9} = \frac{5}{12}.$$

$$\text{Therefore, } \text{var}(\mathbf{X}) = E[\mathbf{X}^2] - (E[\mathbf{X}])^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}.$$

(b) Since the pdf of $\mathbf{Y}$ is the same as the pdf of $\mathbf{X}$, it has the same mean and variance.

$$(c) E[\mathbf{X}\mathbf{Y}] = \int_{u=0}^1 \int_{v=0}^1 uv \cdot 4/3 dv du - \int_{u=0}^{1/2} \int_{v=0}^{1/2} uv \cdot 4/3 dv du = \frac{4}{3} \left[\frac{1}{2} \times \frac{1}{2} - \frac{1}{8} \times \frac{1}{8} \right] = \frac{5}{16} \quad (\text{whoa! why?})$$

Hence, $\text{cov}(\mathbf{X}, \mathbf{Y}) = E[\mathbf{XY}] - E[\mathbf{X}]E[\mathbf{Y}] = \frac{5}{16} - \frac{49}{144} = \frac{-4}{144} = \frac{-1}{36}$ and thus $\rho = \frac{\text{cov}(\mathbf{X}, \mathbf{Y})}{\sqrt{\text{var}(\mathbf{X})\text{var}(\mathbf{Y})}} = \frac{-4}{11}$.

$$\begin{aligned} \text{(d)} \quad E[\min\{\mathbf{X}, \mathbf{Y}\}] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \min(u, v) f_{\mathbf{X}, \mathbf{Y}}(u, v) du dv = \int_{u < v} \int u f_{\mathbf{X}, \mathbf{Y}}(u, v) du dv + \int_{v < u} \int v f_{\mathbf{X}, \mathbf{Y}}(u, v) du dv \\ &= \int_{v=1/2}^1 \int_{u=0}^v u \cdot 4/3 du dv + \int_{u=1/2}^1 \int_{v=0}^u v \cdot 4/3 dv du = 2 \cdot 4/3 \int_{v=1/2}^1 \int_{u=0}^v u du dv = 8/3 \int_{v=1/2}^1 v^2/2 dv = \frac{7}{18}. \end{aligned}$$

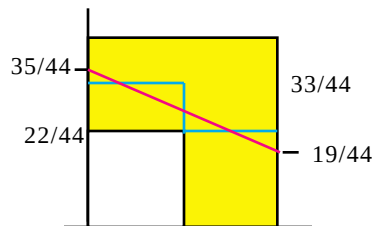
$$\begin{aligned} \text{(e)} \quad E[\max\{\mathbf{X}, \mathbf{Y}\}] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max(u, v) f_{\mathbf{X}, \mathbf{Y}}(u, v) du dv = \int_{u < v} \int v f_{\mathbf{X}, \mathbf{Y}}(u, v) du dv + \int_{v < u} \int u f_{\mathbf{X}, \mathbf{Y}}(u, v) du dv \\ &= \int_{v=1/2}^1 \int_{u=0}^v v \cdot 4/3 du dv + \int_{u=1/2}^1 \int_{v=0}^u u \cdot 4/3 dv du = 2 \cdot 4/3 \int_{v=1/2}^1 \int_{u=0}^v v du dv = 8/3 \int_{v=1/2}^1 v^2 dv = \frac{7}{9}. \end{aligned}$$

(f) Yes, $E[\max\{\mathbf{X}, \mathbf{Y}\}] = \frac{7}{9} > E[\min\{\mathbf{X}, \mathbf{Y}\}] = \frac{7}{18}$ as indeed it should be for any random variables. Since $\max\{\mathbf{X}, \mathbf{Y}\} \geq \min\{\mathbf{X}, \mathbf{Y}\}$, $E[\max\{\mathbf{X}, \mathbf{Y}\}]$ cannot be smaller than $E[\min\{\mathbf{X}, \mathbf{Y}\}]$.

(g) $E[\min\{\mathbf{X}, \mathbf{Y}\}] + E[\max\{\mathbf{X}, \mathbf{Y}\}] = \frac{7}{18} + \frac{7}{9} = \frac{21}{18} = \frac{7}{6} = \frac{7}{12} + \frac{7}{12} = E[\mathbf{X}] + E[\mathbf{Y}]$ as it should be!

(h) $b = \rho \sqrt{\text{var}(\mathbf{Y})/\text{var}(\mathbf{X})} = \rho = \frac{-4}{11}$. $a = E[\mathbf{Y}] - bE[\mathbf{X}] = \frac{7}{12} \left[1 + \frac{4}{11} \right] = \frac{35}{44}$. Hence,

$\tilde{\mathbf{Y}} = \frac{35}{44} - \frac{4\mathbf{X}}{11} = \frac{35 - 16\mathbf{X}}{44}$ which is a straight line from $(0, 35/44)$ to $(1, 19/44)$. The graphs of the best estimate (a step function) and the best linear (straight line) estimate are as shown. Notice that the straight line estimate passes through the mean point $(7/12, 7/12) = (E[\mathbf{X}], E[\mathbf{Y}])$ while the step function estimate does not. This is a characteristic of best linear estimators: if $\mathbf{X}$ equals its mean, the best linear estimate of $\mathbf{Y}$ is the mean of $\mathbf{Y}$. Not always so for nonlinear optimum estimates. Notice also that the straight line estimate is a better approximation to the step function estimate in the interval $(1/2, 1)$ because that is where most of the probability mass is!



The two estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ are the same at $\mathbf{X} = 1/8$ and $\mathbf{X} = 13/16$.

(i) Since $\hat{\mathbf{Y}}$ is a function of $\mathbf{X}$ (not $\mathbf{Y}$!) having value $3/4$ if $\mathbf{X} < 1/2$ and value $1/2$ if $\mathbf{X} > 1/2$, LOTUS allows us to write $E[(\mathbf{Y} - \hat{\mathbf{Y}})] = E[\mathbf{Y}] - E[\hat{\mathbf{Y}}] = \frac{7}{12} - \int_0^{1/2} (3/4) (2/3) du - \int_{1/2}^1 (1/2) (4/3) du = \frac{7}{12} - \frac{1}{4} - \frac{1}{3} = 0$ so that

the average error of the best mean-square estimate is 0. That $E[\mathbf{Y}] = E[\hat{\mathbf{Y}}]$ should be expected from Theorem 4.1 on p. 338 of the book. Similarly,

$$E[(\mathbf{Y} - \tilde{\mathbf{Y}})] = E[\mathbf{Y}] - E[\tilde{\mathbf{Y}}] = E[\mathbf{Y}] - E\left[\frac{35 - 16\mathbf{X}}{44}\right] = E[\mathbf{Y}] - \frac{35}{44} - \frac{16}{44} \times E[\mathbf{X}] = \frac{7}{12} - \frac{35}{44} - \frac{16}{44} \times \frac{7}{12} = 0.$$

$$\text{On the other hand, } E[(\mathbf{Y} - \hat{\mathbf{Y}})^2] = \int_{v=1/2}^1 \int_{u=0}^{1/2} (v-3/4)^2 \cdot 4/3 du dv + \int_{v=0}^1 \int_{u=1/2}^1 (v-1/2)^2 \cdot 4/3 du dv$$

$$= (2/3) \int_{v=1/2}^1 \int_{u=0}^{1/2} (v-3/4)^2 du dv + (2/3) \int_{v=0}^1 \int_{u=1/2}^1 (v-1/2)^2 du dv = \frac{1}{16} = (0.25)^2 \text{ while}$$

$$E[(\mathbf{Y} - \tilde{\mathbf{Y}})^2] = E[(\mathbf{Y} - a - b\mathbf{X})^2] = \text{var}(\mathbf{Y}) \cdot [1 - \rho^2] = \left(\frac{11}{144}\right) \frac{1}{11} - \left(\frac{-4}{11}\right)^2 \frac{1}{528} = \frac{35}{528} \approx (0.257\dots)^2 > \frac{1}{16} = (0.25)^2.$$

Ignoring $\mathbf{X}$ and estimating $\mathbf{Y}$ as $E[\mathbf{Y}] = \frac{7}{12}$ leads to mean-square error $\text{var}(\mathbf{Y}) = \frac{11}{144} \approx (0.276\dots)^2$

(j) Thank goodness!